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On the Normal Series Satisfying Linear Differential Equations

E. Cunningham

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PHILOSOPHICAL TRANSACTIONS.

I. *On the Normal Series Satisfying Linear Differential Equations.*

By E. CUNNINGHAM, *B.A., Fellow of St. John's College, Cambridge.*

Communicated by Dr. H. F. BAKER, *F.R.S.*

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CONTENTS.

1. THE present paper is suggested by that of Dr. H. F. BAKER in the 'Proceedings of the London Mathematical Society,' vol. xxxv., p. 333, "On the Integration of Linear Differential Equations." In that paper a linear ordinary differential equation of order n is considered as derived from a system of n linear simultaneous differential equations

$$\frac{dx_i}{dt} = u_{i1}x_1 + \ldots + u_{in}x_n \quad (i = 1 \ldots n),$$

or, in abbreviated notation,

$$dx/dt = ux,$$

where u is a square matrix of n rows and columns whose elements are functions of t, and x denotes a column of n independent variables.

A symbolic solution of this system is there given and denoted by the symbol $\Omega(u)$. This is a matrix of n rows and columns formed from u as follows :—$Q(\phi)$ is the matrix of which each element is the t-integral from t_0 to t of the corresponding element of ϕ, ϕ being any matrix of n rows and columns ; then

$$\Omega(u) = 1 + Qu + QuQu + QuQuQu \ldots ad\ inf.,$$

where the operator Q affects the whole of the part following it in any term.

Each column of this matrix $\Omega(u)$ gives a set of solutions of the equations

$$dx/dt = ux,$$

and since $\Omega(u) = 1$ for $t = t_0$, these n sets are linearly independent, so that $\Omega(u)$ may be considered as a complete solution of the system.

Part II. of the same paper discusses the form of the matrix $\Omega(u)$ in the neighbourhood of a point at which the elements of the matrix u have poles of the first order, or in the neighbourhood of which the integrals of the original equation are all "*regular*."

It is there shown that if $t = 0$ be such a point, a matrix

$$\phi = \frac{1}{t}\begin{Bmatrix} \theta_1, & c_{21}(t/t_0)^{\theta_1-\theta_2}, & c_{31}(t/t_0)^{\theta_1-\theta_3}\ldots \\ . & \theta_2, & c_{32}(t/t_0)^{\theta_2-\theta_3}\ldots \\ . & . \quad . \quad . \quad . & \theta_3 \\ . & . \quad . \quad . \quad . \quad . & . \quad . \quad . \quad . \end{Bmatrix}$$

can be found, in which all elements to the left of the diagonal are zero, in which $c_{ji} = 0$ unless $\theta_i - \theta_j$ is zero or a positive integer, such that $\Omega(u)$ is of the form

$$G\Omega(\phi)G_0^{-1},$$

where G is a matrix whose elements are converging power series in t, and G_0 is the value of G at $t = t_0$.

The form of ϕ is such as to put in evidence what are known as HAMBURGER's subgroups of integrals associated with the fundamental equation of the singularity ; the method is, in fact, a means of analysing the matrix $\Omega(u)$ into a product of matrices, of which one is expressible in finite terms and shows the nature of the point as a singularity of the solution.

The object of the following investigation is to see how far, under what conditions, and in what form, such an analysis can be effected for equations having poles of a higher order than unity in the elements of the matrix u.

It is known that if in the neighbourhood of infinity the equation is of the form

$$z^{(n)} + \sum_{r=0}^{n-1}\left\{p_r(x) + P_r\left(\frac{1}{x}\right)\right\} z^{(n-r)} = 0,$$

p_r being a polynomial of degree pr, and $P_r(1/x)$ a series of positive integral powers of $1/x$, the equation has a set of *formal* solutions of the form

$$e^{\Omega_r} x^{\sigma_r} \sum_{-\infty}^{0} g_\nu x^\nu, \quad r = 1 \ldots n,$$

where Ω_r is a polynomial of degree $p+1$, provided a certain determinantal equation has its roots all different.

The case in which these roots are not all different is discussed by FABRY ('Thèse, Faculté des Sciences, Paris,' 1885), where he introduces the so-called Subnormal Integrals, viz., integrals of the above form in a variable $x^{1/k}$, k being a positive integer.

The investigation carried out in the following bears the same relation to the discussion of these normal and subnormal integrals that Part II. of the paper quoted at the outset bears to the ordinary analysis of the integrals of an equation in the neighbourhood of a point near which all the integrals are regular.

2. Throughout the discussion the neighbourhood of the point $t = 0$ will be under consideration, the coefficient p_r being supposed to have a pole of order ϖ_r at this point.

Let $p+1$ be the least integer not less than the greatest of the quantities ϖ_r/r. The equation may then be considered as belonging to the more general type

$$z^{(n)} + \sum_{r=1}^{n-1} \frac{P_r(t)}{t^{r(p+1)}} . z^{(n-r)} = 0,$$

where $P_r(t)$ is holomorphic near $t = 0$.

This equation may be reduced to a linear system of simultaneous equations as follows (*vide* 'Proc. Lond. Math. Soc.,' vol. xxxv., p. 344):—

Put $x_1 = z$, $x_2 = t^{p+1} z^{(1)}$, ... $x_{r+1} = t^{r(p+1)} z^{(r)}$, $r = 1, \ldots, n-1$.

The n equations then satisfy the system of n equations

$$\frac{dx}{dt} = \begin{pmatrix} 0, & \frac{1}{t^{p+1}}, & 0, & & & 0 \\ 0, & \frac{p+1}{t}, & \frac{1}{t^{p+1}}, & 0, & & 0 \\ 0, & 0, & 2\frac{p+1}{t}, & \frac{1}{t^{p+1}}, & 0, & 0 \\ & & & & & \\ 0, & \ldots & \ldots & \ldots & (n-2)\frac{p+1}{t}, & \frac{1}{t^{p+1}} \\ \frac{Q_1}{t^{p+1}}, & \ldots & \ldots & \ldots & \frac{Q_{n-1}}{t^{p+1}}, & \frac{Q_n}{t^{p+1}} + (n-1)\frac{p+1}{t} \end{pmatrix} x,$$

where $Q_1 \ldots Q_n$ are series of positive integral powers of t. This system belongs to the more general form

$$\frac{dx}{dt} = ux = \left(\frac{\alpha_{p+1}}{t^{p+1}} + \frac{\alpha_p}{t^p} + \ldots + \frac{\alpha_1}{t} + \alpha_0 + \beta_1 t + \ldots + \beta_r t^r + \ldots\right) x,$$

where $\alpha_{p+1} \ldots \beta_1 \ldots$ are square matrices of constants.

The most general equation of this form will be considered.

If μ be any matrix of constants and $y \equiv \mu x$, the n quantities y satisfy the system

$$\mu^{-1} \frac{dy}{dt} = \frac{dx}{dt} = ux = (u\mu^{-1})\, y,$$

or

$$\frac{dy}{dt} = (\mu u \mu^{-1})\, y.$$

Let μ be now chosen so that $(\mu\alpha_{p+1}\mu^{-1})$ is of canonical form as follows :—(i.) It has zero everywhere save in the diagonal and the $n-1$ places immediately to the right of it; (ii.) The diagonal consists of the roots of the equation $|\alpha_{p+1} - \rho| = 0$, equal roots occupying consecutive places; (iii.) The elements to the right of the diagonal consist of $(\epsilon_1 - 1)$ unities, then a zero, $(\epsilon_2 - 1)$ unities, a zero, and so on ('Proc. Lond. Math. Soc.,' vol. xxxv., p. 352).

Form now the matrices $(\mu\alpha_p\mu^{-1}) \ldots (\mu\beta_r\mu^{-1})$; the equation is then replaced by an equation of exactly similar form, the matrices $\alpha_p \ldots$ being still any matrices whatever, but α_{p+1} being of the canonical form.

3. The equation being denoted by

$$dy/dt = uy,$$

if η be any solution of the equation

$$\text{(A)} \qquad d\eta/dt = u\eta - \eta\chi,$$

χ being an arbitrary matrix, we have

$$\frac{d}{dt}\{\eta\Omega(\chi)\} = \eta\chi\Omega(\chi) + (u\eta - \eta\chi)\,\Omega\chi$$
$$= u\{\eta\Omega(\chi)\},$$

so that $\eta\Omega(\chi)$ is a matrix satisfying the equation in question.

In what follows we are concerned with the form of a solution more than the actual convergence and existence of the same. It is therefore important to notice that if η be a diverging power series formally satisfying equation (A), $\eta\Omega(\chi)$ may be still considered as a formal solution of the original equation, the only condition necessary to secure its actual existence being the convergence of η.

If η be convergent, the solution may be particularized by adding the factor η_0^{-1}, *i.e.*, $\eta\Omega(\chi)\,\eta_0^{-1}$ is the solution reducing to unity at $t = t_0$.

The main investigation to be carried out is that of a simple form for the matrix χ, such that the subsidiary equation (A) may have a formal solution in the form of a matrix whose elements are series of positive integral powers of t, reducing for $t = 0$ to the matrix unity.

4. Owing to the much greater simplicity of the case in which the equation $|\alpha_{p+1} - \rho| = 0$ has all its roots different, it will be treated first separately. The result obtained is as follows :—

A matrix χ can be determined uniquely of the form

$$\frac{\chi_{p+1}}{t^{p+1}} + \frac{\chi_p}{t^p} + \ldots + \frac{\chi_1}{t},$$

where $\chi_{p+1} \ldots \chi_1$ are matrices of constants in which all elements save those in the diagonal are zero, such that there is a formal solution

$$\eta\Omega(\chi),$$

where the matrix η is made up of series of positive integral powers of t—generally diverging—and reducing for $t = 0$ to the matrix unity.

Consider the equation

$$\text{(B)} \qquad \frac{d\eta}{dt} = u\eta - \eta\left(\sum_{r=1}^{p+1} \frac{\chi_r}{t^r}\right),$$

where

$$\chi_r = \begin{pmatrix} \theta_r^{\,1} & 0 & 0 \ldots \\ 0 & \theta_r^{\,2} & 0 \ldots \\ 0 & 0 & \theta_r^{\,3} \ldots \\ . & . & . \quad . \quad . \quad \theta_r^{\,n} \end{pmatrix} \qquad r = 1, \ldots, p+1.$$

The roots of $|\alpha_{p+1} - \rho| = 0$ being unequal, the matrix α_{p+1} will have zero elements except in the diagonal; the diagonal elements will be $\rho_1, \rho_2, \ldots \rho_n$, the roots of the equation.

If the equation (B) is satisfied by the matrix

$$\eta = (x, y, z \ldots),$$

where $x, y, z \ldots$ denote columns of elements of the form

$$x = x_0 + x_1 t + x_2 t^2 + \ldots,$$

$$y = y_0 + y_1 t + \ldots,$$

the coefficients $x_r, y_r \ldots$ being columns of constants $x_r^{\,0}, x_r^{\,1}, x_r^{\,2}$, &c., these constants satisfy the following equations :—

$$\text{X.} \qquad \begin{aligned} &(\alpha_{p+1} - \theta^1{}_{p+1})\, x_0 = 0, \\ &(\alpha_{p+1} - \theta^1{}_{p+1})\, x_1 + (\alpha_p - \theta_p^{\,1})\, x_0 = 0, \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

$$(\alpha_{p+1}-\theta^1{}_{p+1})x_p+\ldots \quad +(\alpha_1-\theta_1{}^1)x_0 = 0,$$
$$(\alpha_{p+1}-\theta^1{}_{p+1})x_{p+1}+\ldots+(\alpha_1-\theta_1{}^1-1)x_1+\alpha_0 x_0 = 0,$$
$$\cdots\cdots\cdots$$
$$(\alpha_{p+1}-\theta^1{}_{p+1})x_{p+r}+\ldots+(\alpha_1-\theta_1{}^1-r)x_r+\alpha_0 x_{r-1}+\beta_1 x_{r-2}+\ldots+\beta_{r-1}x_0 = 0,$$
$$\cdots\cdots\cdots$$

A precisely similar set of equations gives the relations connecting the constants y and $\theta^2{}_{p+1}\ldots\theta_1{}^2$.

The equations X just written *determine uniquely a set of values for* $\theta^1{}_{p+1}\ldots\theta_1{}^1$ *and the coefficients* $x_0\ldots$.

The first of these equations gives

$$(\rho_1-\theta^1{}_{p+1})x_0{}^1 = 0, \quad (\rho_2-\theta^1{}_{p+1})x_0{}^2 = 0.$$

Since x_0 is to be equal to $\begin{pmatrix}1\\ \vdots \\ 0\end{pmatrix}$ we must have $\theta^1{}_{p+1} = \rho_1$, and these equations are then satisfied.

Similarly the first of the y equations with $y_0 = \begin{pmatrix}0\\1\\ \vdots \\ 0\end{pmatrix}$ gives $\theta^2{}_{p+1} = \rho_2$; and so for the other columns.

The second of the equations X written more fully gives

$$\alpha_p{}^{11}-\theta_p{}^1 = 0, \quad (\rho_r-\rho_1)x_1{}^r+\alpha_p{}^{1r} = 0, \quad r = 2, \ldots, n.$$

These then determine x_1 save for its first element, in place of which a unique value is given for $\theta_p{}^1$.

The third equation X in full gives

$$\alpha_p{}^{21}x_1{}^2+\ldots+\alpha_p{}^{n1}x_1{}^n+\alpha^{11}{}_{p-1}-\theta^1{}_{p-1} = 0,$$
$$(\rho_r-\rho_1)x_2{}^r+\alpha_p{}^{1r}x_1{}^1+\alpha_p{}^{2r}x_1{}^2\ldots+(\alpha_p{}^{rr}-\theta_p{}^1)x_1{}^r+\ldots+\alpha_p{}^{nr}x_1{}^n+\alpha^{1r}{}_{p-1} = 0, \quad r = (2, \ldots, n).$$

Of these, the first gives $\theta^1{}_{p-1}$, while the following determine x_2 save for its first element, but only in terms of the yet undetermined $x_1{}^1$.

Of the next group, the first equation is

$$\alpha_p{}^{21}x_2{}^2\ldots+\alpha_p{}^{n1}x_2{}^n+(\alpha^{11}{}_{p-1}-\theta^1{}_{p-1})x_1{}^1+\alpha^{21}{}_{p-1}x_1{}^2+\ldots+\alpha^{n1}{}_{p-1}x_1{}^n+\alpha^{11}{}_{p-2}-\theta^1{}_{p-2} = 0.$$

This equation apparently involves the unknown $x_1{}^1$ explicitly, and also, in $x_2{}^2\ldots x_2{}^n$, implicitly.

But the whole coefficient of $x_1{}^1$ is

$$\sum_2^n \alpha_p{}^{k1}\left(-\frac{\alpha_p{}^{1k}}{\rho_{k-\rho_1}}\right)-\sum_2^n \alpha_p{}^{k1}x_1{}^k$$
$$= \sum_2^n \alpha_p{}^{k1}x_1{}^k-\sum_2^n \alpha_p{}^{k1}x_1{}^k = 0,$$

so that $\theta^1{}_{p-2}$ is given independently of $x_1{}^1$.

The remaining equations of this group give x_3 except for x_3^1, in terms of x_1^1 and x_2^1.

Proceeding in this way as far as the $(p+1)^{\text{th}}$ equation of X, $x_1 \ldots x_p$ are all found except for their first elements, while the first elements of these equations give $\theta^1_{p+1} \ldots \theta_1^1$, θ^1_{p+r} being not *a priori* independent of $x_1^1 \ldots x_r^1$ hitherto unknown.

It has been shown above, however, that the determination of θ^1_{p-1} does not require a knowledge of x_1^1.

In general, in fact, θ^1_{p-r} *is given independently of* $x_1^1 \ldots x_r^1$.

To prove this, the way in which x_1^1 enters into x^k_{r+1} will be first considered. This may be stated as follows :—

The coefficient of x_1^1 in x^k_{r+1} is equal to that part of x_r^k which is independent of $x_1^1 \ldots x^1_{r-1}$.

For $r = 1$ this is at once seen by writing down the equations

$$(\rho_2 - \rho_1) x_1^k + \alpha_p^{1k} = 0,$$
$$(\rho_k - \rho_1) x_2^k + \alpha_p^{1k} x_1^1 + \ldots + \alpha_p^{nk} x_1^n + \alpha^{1k}_{p-1} = 0.$$

In general the equation for x^k_{r+1} is

$$-(\rho_k - \rho_1) x^k_{r+1} = \sum_{s=1}^{r} \{\alpha^{1k}_{p-r+s} x_s^1 + \ldots + (\alpha^{kk}_{p-r+s} - \theta^1_{p-r+s}) x_s^k \ldots + \alpha^{nk}_{p-r+s} x_s^n\} + \alpha^{1k}_{p-r}.$$

Assuming the statement above to be true for 1, $2 \ldots r$, and that $\theta_p, \ldots, \theta_{p-r+1}$ are independent of $x_1^1 \ldots x^1_{r-1}$, the above equation shows that the coefficient of x_1^1 in $-(\rho_k - \rho_1) x^k_{r+1}$ is the part independent of $x_1^1, \ldots, x_r^1$ in

$$\alpha^{1k}_{p-r+1} + \sum_{s=1}^{r-1} \{\alpha^{1k}_{p-r+s+1} x_s^1 + \ldots + \alpha^{nk}_{p-r+s+1} x_s^n\},$$

i.e., in

$$-(\rho_k - \rho_1) x_r^k,$$

so that under the above assumptions the statement holds for 1, $2 \ldots$, $r+1$.

Also, under the same assumptions, from the equation giving θ^1_{p-r}, viz.,

$$\begin{aligned} &\alpha_p^{21} x_r^2 + \ldots + \alpha_p^{n1} x_r^n \\ &\quad + (\alpha^{11}_{p-1} - \theta^1_{p-1}) x^1_{r-1} + \alpha^{21}_{p-1} x^2_{r-1} + \ldots + \alpha^{n1}_{p-1} x^n_{r-1} + \ldots \\ &\quad + (\alpha^{11}_{p-r+1} - \theta^1_{p-r+1}) x_1^1 + \ldots + \alpha^{n1}_{p-r+1} x_1^n + (\alpha^{11}_{p-r} - \theta^1_{p-r}) = 0, \end{aligned}$$

we deduce that the coefficient of x_1^1 in θ^1_{p-r} is the part independent of x_1^1 in

$$\begin{aligned} &\alpha_p^{21} x^2_{r-1} + \ldots + \alpha_p^{n1} x^n_{r-1} \\ &\quad + \alpha^{21}_{p-1} x^2_{r-2} + \ldots + \alpha^{n1}_{p-1} x^n_{r-2} + \ldots \\ &\quad + \alpha^{21}_{p-r+2} x_1^2 + \ldots + \alpha^{n1}_{p-r+2} x_1^n + \alpha^{11}_{p-r+1} - \theta^1_{p-r+1}. \end{aligned}$$

This expression differs only from the left-hand member of the equation for θ^1_{p-r+1} by multiples of $x_1^1 \ldots x^1_{r-1}$, and therefore, on the assumption that this equation gives θ^1_{p-1} independently of $x_1^1 \ldots$, the part independent of these quantities in the above expression must, when θ^1_{p-r+1} is determined, vanish, so that θ^1_{p-r} is independent of x_1^1.

Now the way in which the successive equations follow one another shows that the coefficient of x_1^1 in θ_k^1 is equal to that of x_r^1 in θ_{k+r-1}.

Thus θ^1_{p-r+1} being independent of x_1^1, θ^1_{p-r} is independent of x_2^1, and in general, $\theta^1_{p-r+k}(k = 1 \ldots r-1)$ being all independent of x_1^1, θ^1_{p-r} does not contain $x_1^1 \ldots x^1_{r-1}$.

Thus, if the assumptions made on p. 7 are satisfied for any particular value of r less than p, they are satisfied for a value of r one greater than that value.

For $r = 1$ the statements have been justified, and it follows therefore that $\theta^1_{p+1} \ldots \theta_1^1$ *are all determined uniquely* without the knowledge of x_1^1, $x_2^1 \ldots$ from the first $(p+1)$ of the equations X, and by the same equations $x_1 \ldots x_p$ are found, except for their first elements, the expressions obtained containing those first elements.

5. Consider now the $(p+2)^{\text{th}}$ equation X in regard to its first element.

As before, this will be independent of $x_p^1 \ldots x_2^1$; but on account of the extra term arising from $d\eta/dt$, which now enters for the first time, the coefficient of x_1^1 is not zero. It is, in fact, -1.

Thus, the quantities $\theta_1^1 \ldots \theta^1_{p+1}$ being now known, this equation gives x_1^1.

Similarly, the next group's first member will contain the term $-2x_2^1$ but will not contain $x_3^1 \ldots x^1_{p+1}$, and will therefore give x_2^1 after x_1^1 is found.

Thus all the elements x_k^1 are determined successively, and returning to the expressions for $x_k^r \,(r > 1)$ in terms of these and substituting the values so found, all these are given also.

The equations for the columns y, z, &c., being treated in the same way, give the corresponding θ's uniquely, and also the coefficients in the series of which these columns are composed.

Thus it is shown that when the "characteristic equation" $|\alpha_{p+1} - \rho| = 0$ has its roots all different, the equation

$$dy/dt = uy,$$

where

$$u = \frac{\alpha_{p+1}}{t^{p+1}} + \ldots + \frac{\alpha_1}{t} + \alpha_0 + \beta_1 t + \ldots,$$

α_{p+1} being in its canonical form, possesses a unique formal solution in the form

$$\eta \Omega \left(\frac{\alpha_{p+1}}{t^{p+1}} + \frac{\chi_p}{t^p} + \ldots + \frac{\chi_1}{t} \right),$$

where the elements of $\chi_p \ldots \chi_1$ not in the diagonal are zero, and the elements of η are power series in t, reducing for $t = 0$ to the matrix unity.

The matrix $\Omega\left(\frac{\alpha_{p+1}}{t^{p+1}} + \ldots + \frac{\chi_1}{t}\right)$ can at once be written in the form ω/ω_0, where ω is a matrix whose non-diagonal elements are zero, and whose k^{th} diagonal element is

$$e^{-\frac{\theta^k{}_{p+1}}{p \,.\, t^p} - \frac{\theta_p{}^k}{(p-1)\, t^{p-1}} \cdots - \frac{\theta_2{}^k}{t}} \,.\, t^{\theta_1{}^k},$$

and ω_0 is the value of ω at $t = t_0$.

If the series η happen to be convergent, the solution which reduces to unity at $t = t_0$ can at once be written in the form $\eta\omega\omega_0^{-1}\eta_0^{-1}$.

Applied to the system formed from a particular linear ordinary equation we have at once the result referred to on p. 3 (v. SCHLESINGER, 'Lin. Diff.-Gleichungen,' vol. I., pp. 341 ff.*).

As a simple example of the application of the method we may take the well-known equation

$$\frac{d^2w}{dz^2} = \frac{\alpha + \beta z + \gamma z^2}{z^4} w.$$

Putting $w_1 = w$ and $w_2 = z^2\, dw/dt$

$$dw_1/dt = \frac{w_2}{z^2}, \quad dw_2/dt = \left(\frac{\alpha}{z^2} + \frac{\beta}{z} + \gamma\right) w_1 + \frac{2}{z} w_2,$$

which in matrix notation is

$$dw/dt = \left\{\begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}\frac{1}{z^2} + \begin{pmatrix} 0 & 0 \\ \beta & 2 \end{pmatrix}\frac{1}{z} + \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}\right\} w.$$

The equation $\begin{vmatrix} -\rho & 1 \\ \alpha & -\rho \end{vmatrix} = 0$ gives $\rho = \pm\sqrt{\alpha}$, and the matrix μ needed to transform the first matrix to canonical form is $\begin{pmatrix} \sqrt{\alpha} & 1 \\ -\sqrt{\alpha} & 1 \end{pmatrix}$, so that the equation is transformed to

$$du/dt = \left[\frac{1}{z^2}\begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix} + \frac{1}{z}\begin{pmatrix} 1 + \frac{\beta}{2\sqrt{\alpha}} & 1 - \frac{\beta}{2\sqrt{\alpha}} \\ 1 + \frac{\beta}{2\sqrt{\alpha}} & 1 - \frac{\beta}{2\sqrt{\alpha}} \end{pmatrix} + \frac{\gamma}{2\sqrt{\alpha}}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\right] u,$$

$$= \mathrm{V}u, \text{ say.}$$

The subsidiary equations are

$$d\eta/dt = \mathrm{V}\eta - \eta\left[\frac{1}{z^2}\begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix} + \frac{1}{z}\begin{pmatrix} 1 + \frac{1}{2}\lambda, & 0 \\ 0, & 1 - \frac{1}{2}\lambda \end{pmatrix}\right], \text{ where } \lambda = \frac{\beta}{\sqrt{\alpha}},$$

which give as the general relations connecting the coefficients of the first column of η, putting $p = 1 + \frac{1}{2}\lambda$, $q = 1 - \frac{1}{2}\lambda$,

$$\left.\begin{aligned} -nx_n^1 + qx_n^2 + \frac{\gamma}{2\sqrt{\alpha}}(x^1_{n-1} - x^2_{n-1}) = 0 \\ -2\sqrt{\alpha}x^2_{n+1} + px_n^1 + (q - p - n)\, x_n^2 + \frac{\gamma}{2\sqrt{\alpha}}(x^1_{n-1} - x^2_{n-1}) = 0 \end{aligned}\right\} \quad \text{(i.).}$$

Hence

$$-2\sqrt{\alpha}x^2_{n+1} + (p + n)(x_n^1 - x_n^2) = 0,$$

and therefore

$$-2\sqrt{\alpha}x_n^2 + (p + n - 1)(x^1_{n-1} - x^2_{n-1}) = 0 \quad \text{(ii.),}$$

which with the first equation gives

$$2\sqrt{\alpha}\,.\,n(x_n^1 - x_n^2) = (x^1_{n-1} - x^2_{n-1})\{\gamma + (q - n)(p + n - 1\}.$$

Thus a recurrence formula is established for the quantities $x_n^1 - x_n^2$ in terms of which x^1_{n+1} and x^2_{n+1} can be at once expressed.

* With reference to SCHLESINGER'S demonstration of this result, see a note by the author in the 'Messenger of Mathematics,' January, 1905.

The series for x will terminate if for any value of n

$$\gamma+(q-n)(p+n-1)=0,$$

i.e., if

$$-\gamma+(\tfrac{1}{2}\lambda+n)(\tfrac{1}{2}\lambda+n-1)=0,$$

i.e., if

$$(\lambda+2n-1)^2-(4\gamma+1)=0$$

for some value of n.

The series for y will similarly terminate if

$$(-\lambda+2m-1)^2-(4\gamma+1)$$

vanish for some value of n.

Both these are certainly satisfied if

$$\lambda=q \quad \text{and} \quad 4\gamma+1=p^2,$$

where p, q are any integers of which one is odd and the other even.

6. We pass now to the case where the characteristic equation $|\alpha_{p+1}-\rho|=0$ has its roots not all unequal, and the analysis becomes a good deal more intricate with the less simple canonical form of the matrix α_{p+1} as stated on p. 4. It will be remembered that the numbers ϵ_1, $\epsilon_2\ldots$ there used are the powers of $(\rho_1-\rho)$ in the elementary divisors of $|\alpha_{p+1}-\rho|$ with respect to the root ρ_1 of this equation of multiplicity l. In the case of the system obtained on p. 4 from a single equation of order n, we may prove that $\epsilon_1=l$, $\epsilon_2=\epsilon_3=\ldots=0$.

For the matrix $(\alpha_{p+1}-\rho)$ is of the form

$$\left(\begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & 0 & 1 \\ a & b & . & . & . & . & k \end{pmatrix} -\rho\right).$$

The minor of the quantity "k" in the determinant of this matrix is simply $(-\rho)^{n-1}$. Thus the elementary divisors are certainly merely unity with respect to any non-zero roots.

If there be a multiple zero root, however, since the minor of "a" is unity, the elementary divisors with respect to this root are all simply ρ^0.

Thus for such a system we have for each multiple root $\epsilon_2=\epsilon_3\ldots=0$; so that in the canonical form of α_{p+1}, if

$$\rho_i=\rho_{i+1}, \quad \alpha_{p+1}^{i+1,\,i}=1,$$

and if

$$\rho_i\neq\rho_{i+1}, \quad \alpha_{p+1}^{i+1.\,i}=0.$$

Such systems being by far the most important in practice and also considerably simpler to work out, the full discussion will be restricted to systems of this type. It may be pointed out that the most general system can be solved by means of the solution of systems of this more restricted type, for from the general system

$\frac{dx}{dt} = \frac{u}{t^{p+1}}\,x$, where u is a power series in t, a linear equation of order n and rank p near $t = 0$ can be obtained for each row of the matrix x, and this equation can be solved by the solution of a linear system of the restricted type in question.

Of the matrix χ to be used here, the following properties will be presupposed :—

(i.) It is to be of the form $\frac{\chi_{p+1}}{t^{p+1}}+\frac{\chi_p}{t^p}+\ldots+\frac{\chi_1}{t}$, where each of the matrices $\chi_1 \ldots \chi_{p+1}$ has all elements to the left of the diagonal zero.

(ii.) The diagonal elements of these matrices are to be numerical constants denoted as before by θ_r^s ($r = 1, \ldots, p+1$; $s = 1, \ldots, n$).

(iii.) All the other non-zero elements of $\chi_2\chi_3\ldots\chi_{p+1}$ are to be constants, while the other elements of χ_1 may contain t, but only to positive integral powers (*cf.* the matrix χ in Dr. BAKER's paper, *loc. cit.*).

8. As before, the matrix η, which is a formal solution of the subsidiary equation

$$d\eta/dt = u\eta - \eta\chi,$$

will be supposed to be formed of the columns

$$x = x_0 + x_1 t + \ldots,$$

$$y = y_0 + y_1 t + \ldots,$$

and the equations for the coefficients $x_r y_r \ldots$ are the same as the equations X (p. 5).

But the detailed form of these equations is quite different. The first of them $(\alpha_{p+1} - \theta^1{}_{p+1})\,x_0 = 0$ is still satisfied by

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \quad \theta^1{}_{p+1} = \rho_1.$$

Supposing now ρ_1 to be a root of multiplicity ϵ_1, the second equation X is in full

$$x_1{}^2 + \alpha_p{}^{11} - \theta_p{}^1 = 0,$$

$$x_1{}^3 + \alpha_p{}^{12} \qquad = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\alpha_p{}^{1\epsilon_1} = 0,$$

$$(\rho_{\epsilon_1+1} - \rho_1)\,x_1{}^{\epsilon_1+1} + x_1{}^{\epsilon_1+2} + \alpha_p{}^{1,\epsilon_1+1} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(\rho_{\epsilon_1+\epsilon_2} - \rho_1)\,x_1{}^{\epsilon_1+\epsilon_2} + \alpha_p{}^{1,\epsilon_1+\epsilon_2} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where

$$\rho_{\epsilon_1+1} = \ldots = \rho_{\epsilon_1+\epsilon_2} \neq \rho_1.$$

These equations manifestly determine x_1 except for its first and second elements, the second being known as soon as $\theta_p^{\,1}$ is.

We are also at once faced with a condition necessary for the possibility of the solution under the assumptions made as to the form of χ, viz. :—

$$\alpha_p^{\,1\epsilon_1} = 0.$$

This condition arises from the ϵ_1^{th} equation of the set, and as, in the ensuing discussion, the ϵ_1^{th} equation of each set is most important we shall here introduce a notation for it, viz., X_r will stand for the ϵ_1^{th} equation of the $(r+1)^{\text{th}}$ set; *i.e.*, of the set

$$(\alpha_{p+1} - \theta^1_{\ p+1})\, x_r + \ldots = 0.$$

This equation will not contain any element of x_r.

A similar notation will be adopted for the equations Y, Z... for the coefficients in the other columns of η.

If the second element of the first row of χ_{p+1} be c_{21}, the equations Y are

$$(\alpha_{p+1} - \theta^2_{\ p+1})\, y_0 - c_{21} x_0 = 0,$$
$$(\alpha_{p+1} - \theta^2_{\ p+1})\, y_1 + (\alpha_p - \theta_p)\, y_0 - c_{21} x_1 = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Of these the first is satisfied by

$$y_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \quad \theta^2_{\ p+1} = \rho_2 = \rho_1,$$

provided we take $c_{21} = 1 =$ corresponding element in α_{p+1}.

Considering each of the columns in succession we have thus, with $\eta_0 = 1$, $\chi_{p+1} = \alpha_{p+1}$.

The second of the equations Y gives

$$y_1^{\,2} + \alpha_p^{\,21} - x_1^{\,1} = 0,$$
$$y_1^{\,3} + \alpha_p^{\,22} - \theta_p^{\,2} - x_1^{\,2} = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\alpha_p^{\,2\epsilon_1} - x_1^{\,\epsilon_1} = 0,$$
$$(\rho_{\epsilon_1+1} - \rho_1)\, y_1^{\,\epsilon_1+1} + \alpha_p^{\,2\epsilon_1+1} - x_1^{\,\epsilon_1+1} = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

which, when x_1 is known, determine y_1 save for its first element, and its third until $\theta_p^{\,2}$ is known.

The exceptional equation Y_1, $\alpha_p^{\,2\epsilon_1} - x_1^{\,\epsilon_1} = 0$, gives us again a necessary condition for the possibility of the solution in view, $\alpha_p^{\,2,\epsilon_1} + \alpha_p^{\,1,\epsilon_1-1} = 0$.

Similarly the equation Z_1 gives

$$0 = \alpha_p^{3,\epsilon_1} - y_1^{\epsilon_1} = \alpha_p^{3,\epsilon_1} + \alpha_p^{2,\epsilon_1-1} - x_1^{\epsilon_1-1} = \alpha_p^{3,\epsilon_1} + \alpha_p^{2,\epsilon_1-1} + \alpha_p^{1,\epsilon_1-2},$$

and so on for the equations for each of the first ϵ_1 columns of η.

For the $(\epsilon_1+1)^{\text{th}}$ column, however, the non-diagonal term of $\chi_{p+1} = 0$ and the equations for this column do not contain the elements of the preceding column.

In fact, the ϵ_2 columns beginning with the $(\epsilon_1+1)^{\text{th}}$ form a group related to one another in just the same manner as the first ϵ_1 are. We obtain from them, as from the latter, the conditions

$$\alpha_p^{\epsilon_1+1,\,\epsilon_1+\epsilon_2} = 0,$$
$$\alpha_p^{\epsilon_1+1,\,\epsilon_1+\epsilon_2-1} + \alpha_p^{\epsilon_1+2,\,\epsilon_1+\epsilon_2} = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\alpha_p^{\epsilon_1+1,\,\epsilon_1+1} + \ldots + \alpha_p^{\epsilon_1+\epsilon_2,\,\epsilon_1+\epsilon_2} = 0,$$

and so for the columns associated with each group of equal roots. These and other conditions which arise in the course of the investigation will be called "equations of condition." Supposing those already found to be satisfied, we may return to the solution of the equations X, and of these the following statements are to be proved:—

I. The first ϵ_1-1 equations of the $(r+1)^{\text{th}}$ set determine $x_r^2 \ldots x_r^{\epsilon_1}$ in terms of $x_1^1 \ldots x^1_{r-1}$ and $\theta^1_{p+1} \ldots \theta^1_{p-r+1}$; the equations from the $(\epsilon_1+1)^{\text{th}}$ onwards give $x_r^{\epsilon_1+1} \ldots x_r^n$ in terms of the same quantities.

II. When the values thus found are substituted in the equation X_{r+1}, the resulting left-hand member is *independent of the undetermined quantities* $x_1^1 \ldots x_r^1$, $\theta_p^1 \ldots \theta^1_{p-r}$ for all values of r up to (ϵ_2-2), but for $r = \epsilon_1-1$ it is independent of all save θ_p^1; in fact, the equation X_{ϵ_1} *is an algebraic equation of degree* ϵ_1 *for* θ_p^1 and contains no other undetermined quantity.

III. Supposing one root of this to be chosen for the value of θ_p^1, and the equations Y to be treated in the same way, Y_{ϵ_1-1} will be an equation of degree (ϵ_1-1) in θ_p^2, *whose roots are exactly the remaining roots of* X_{ϵ_1}.

IV. Similarly Z_{ϵ_1-2} furnishes an equation of degree ϵ_1-2 for θ_p^3, whose roots are the remaining roots of Y_{ϵ_1-1}, and therefore of X_{ϵ_1}, and so on.

Thus $\theta_p^1 \ldots \theta_p^{\epsilon_1}$ are the roots of the equation X_{ϵ_1}.

V. The values of $\theta_{p-1} \ldots$ subsequently obtained in association with each of these roots will be the same in whatever order they are taken.

Of these I. does not require proof.

With regard to II., the proof that the equations do not contain the undetermined x's follows exactly the same lines as the corresponding proof when the characteristic equation has its roots all different (*vide* pp. 7–8).

The proof that they do not contain $\theta_p^1 \ldots$ until $r = \epsilon_1-1$ requires considerations of a different kind involving the equations X, Y, Z... simultaneously.

9. Consider the system of equations X^1 derived from X by changing θ_p^1 into θ_p^2 and x into x^1, viz. :—

$$X^1. \qquad (\alpha_{p+1}-\theta^1_{p+1})x_0^1 = 0,$$
$$(\alpha_{p+1}-\theta^1_{p+1})x_1^1+(\alpha_p-\theta_p^2)x_0^1 = 0,$$
$$\cdots\cdots\cdots\cdots,$$

and let these be treated in exactly the same way as the equations X, the undetermined elements of x^1 being supposed the same as those of x.

From the two sets of equations X, X^1 let a new set be formed by subtracting corresponding members of X and X^1 and dividing each remainder by $\theta_p^1-\theta_p^2$, and let this new system be denoted by

$$\Delta_p X = \frac{X-X^1}{\theta_p^1-\theta_p^2} = 0.$$

The expressions for x^1 obtained from X^1 in terms of $\theta^1_{p+1}\ldots$ will be identical with those obtained from X in terms of the same with θ_p^1 changed into θ_p^2.

Let $\Delta_p x$ denote the expression $\dfrac{x-x^1}{\theta_p^1-\theta_p^2}\cdot$

Then $\Delta_p X$ is the system of equations

$$0 = 0, \quad (\alpha_{p+1}-\theta^1_{p+1})\Delta_p x_1 - x_0 = 0,$$
$$(\alpha_{p+1}-\theta^1_{p+1})\Delta_p x_2+(\alpha_p-\theta_p^2)\Delta_p x_1 - x_1 = 0,$$
$$\cdots\cdots\cdots\cdots,$$
$$(\alpha_{p+1}-\theta^1_{p+1})\Delta_p x_{r+1}+(\alpha_p-\theta_p^2)\Delta_p x_r+\ldots-x_r = 0,$$
$$\cdots\cdots\cdots\cdots$$

Further
$$\Delta_p(x_1) = \begin{pmatrix}0\\1\\0\\ \vdots\end{pmatrix} = y_0, \quad \text{and} \quad \theta^1_{p+1} = \theta^2_{p+1}.$$

Thus these equations are identical in form with the equations Y, except that θ^2_{p-k} is replaced by θ^1_{p-k}, $k>0$. Thus if from Y the y's be calculated as in I., p. 13, and from $\Delta_p X$ the quantities $\Delta_p x$ be similarly determined, the only difference between Δx_{r+1} and y_r will be in the substitution of $x_1^1\ldots$ for $y_1^1\ldots$ and θ^1_{p-k} for θ^2_{p-k}, $k>0$, and (θ_1^1+1) for θ_1^2.

Thus if we substitute the values of $\Delta_p x$ thus found in $\Delta_p X_{r+1}$ the result will differ from Y_r only by the same substitutions.

In a similar way, denoting by $\Delta_p Y$ the difference equations

$$\frac{Y-Y^1}{\theta_p^2-\theta_p^3},$$

$\Delta_p Y$ will differ from Z_{r-1} only by the substitution of $y_1^1\ldots$ for z_1^1, θ^2_{p-1} for $\theta^3_{p-1}\ldots$, and θ_1^2+1 for θ_1^3, and so for the remainder of the first ϵ_1 columns.

In exactly the same way if X^2 denote the system derived from X by the change of θ^1_{p-1} into θ^2_{p-1}, and the equations

$$\Delta_{p-1}X = \frac{X - X^2}{\theta^1_{p-1} - \theta^2_{p-1}}$$

be formed, the quantities $\Delta_{p-1}x_{r+2}$ differ from y_r only by the substitution of θ_p^1 for θ_p^2, θ^1_{p-2} for $\theta^2_{p-2}\ldots$ and θ_1^1+2 for θ_1^2. The same operator Δ_{p-1} applied to the equations Y connects the columns y and z, and so on.

Similarly, operators $\Delta_{p-2}\ldots\Delta_2$ may be defined.

Lastly, an operator Δ_1 may be defined so that the equation $\Delta_1 X$ is $\dfrac{X - X^{(p)}}{\theta_1^1 - \theta_1^2 + p}$, where $X^{(p)}$ denotes the equations X with $\theta_1^2 - p$ substituted for θ_1^1. Then the equation Y_r will, when $y_1^1,\ldots$ are replaced by $x_1^1,\ldots$ and $\theta_p^2,\ldots,\ \theta_2^2$ by $\theta_p^1,\ldots,\ \theta_2^1$, become the equation $\Delta_p X_{p+r}$.

Consider now the equation Z_1; it is independent of θ_p^3, $z_1^1,\ldots$, and therefore reduces to a simple numerical constant which must be zero (p. 13).

But Y is a polynomial in θ_p^2. It can therefore be only of the first degree, since $\Delta_p Y_2$ is independent of θ_p^3; is, in fact, the same as Z_1, viz., zero, so that Y_2 does not contain θ_p^2. It must then, like Z_1, be only a constant, and must therefore vanish identically, so that $Y_2 = 0$ gives a further "equation of condition."

Hence again X_3 cannot contain θ_p^1, and the operator Δ_{p-1} connecting it with Y_1 shows that it cannot contain θ^1_{p-1}. Thus X_3 again must be a vanishing constant, giving another "equation of condition."

Similarly, starting with the corresponding equation of the fourth column, we find $\Delta_p X_4 \equiv 0$, so that X_4 must be independent of θ_p^1. Also $\Delta_{p-1}X_4$ will vanish identically, so that X_4 is independent of θ^1_{p-1}, and similarly it is independent of $\theta^1_{p-2}\ldots$.

Thus if $\epsilon_1 \geqq 4$, X_4 reduces to a mere constant which, as before, must vanish.

The process may clearly be carried on as far as the ϵ_1th column, so that the equations $X_1\ldots X_{\epsilon_1-1}$ all give equations of condition, as do also $Y_1\ldots Y_{\epsilon_1-2}$, $Z_1\ldots Z_{\epsilon_1-3}$, etc. Starting now from the second equation of the ϵ_1th column,

$$\alpha_p^{\epsilon_1\epsilon_1} - \theta_p^{\epsilon_1} - \xi_1^{\epsilon_1} = 0,$$

where ξ denotes the ϵ_1th column of η, it follows that the third equation from (ϵ_1-1)th column must be a quadratic in $\theta_{p-1}^{\epsilon_1-1}$, independent of ξ_1^1, $\{=\phi_2(\theta_p^{\epsilon_1-1})$ say$\}$, and such that

$$\frac{\phi_2(\theta_p^{\epsilon_1-1}) - \phi_2(\theta_p^{\epsilon_1})}{\theta_p^{\epsilon_1-1} - \theta_p^{\epsilon_1}} \equiv \alpha_p^{\epsilon_1\epsilon_1} - \theta_p^{\epsilon_1} - \xi^{\epsilon_1} = 0.$$

Thus if $\theta_p^{\epsilon_1-1}$ is one root of $\phi_2 = 0$, $\theta_p^{\epsilon_1}$ is the other.

Similarly, the fourth equation from the preceding column must give a cubic for $\theta_p^{\epsilon_1-2}$, $(\phi_3 = 0)$, such that

$$\frac{\phi_3(\theta_p^{\epsilon_1-2})-\phi_3(\theta_p^{\epsilon_1-1})}{\theta_p^{\epsilon_1-2}-\theta_p^{\epsilon_1-1}} \equiv \phi_2(\theta_p^{\epsilon_1-1}).$$

Thus $\theta_p^{\epsilon_1-2}$, $\theta_p^{\epsilon_1-1}$, $\theta_p^{\epsilon_1}$ are the roots of $\phi_3 = 0$.

Eventually the first column gives an equation of degree ϵ_1 for θ_p^1 (viz., X_{ϵ_1}), of which the roots in any order are a possible set of values for $\theta_p^1 \ldots \theta_p^{\epsilon_1}$. Calling this equation $\phi(\theta) = 0$, and denoting its roots in some assigned order by $\sigma_1, \sigma_2, \ldots \sigma_{\epsilon_1}$, let us consider the values determined for $\theta^1_{p-1} \ldots$ by taking $\theta_p^1 = \sigma_1 \ldots$ and $\theta_p^{\epsilon_1} = \sigma_{\epsilon_1}$.

10. Again, as prior in order of simplicity, let the case in which the roots of $\phi(\theta)$ are all different be taken first.

It has been shown above that the equation Y_r, when $x_1^1 \ldots$ are substituted for $y_1^1 \ldots$ and θ_p^1, $\theta^1_{p-2} \ldots$ for θ_p^2, $\theta^2_{p-2} \ldots$, becomes identical with $\Delta_{p-1} X_{r+2}$.

Now, Y_{ϵ_1-1} is merely a polynomial in θ_p^2, independent of $y_1^1 \ldots$ and $\theta^2_{p-1} \ldots$, and vanishing for $\theta_p^2 = \sigma_2, \sigma_3 \ldots \sigma_{\epsilon_1}$.

Let
$$Y_{\epsilon_1-1} = \phi_{\epsilon_1-1}(\theta_p^2).$$

Thus X_{ϵ_1+1} is linear in θ^1_{p-1} the coefficient of the same being $\phi_{\epsilon_1-1}(\theta_p^1)$; the part independent of θ^1_{p-1} contains only θ_p^1, which is now a determinate quantity.

If the roots of $\phi_{\epsilon_1}(\theta_p)$ are all unequal, $\phi_{\epsilon_1-1}(\theta_p^1) \neq 0$ and θ^1_{p-1} is given uniquely; and similarly $Y_{\epsilon_1} = 0$ is a linear equation for θ^2_{p-1} in terms of θ_p^2 and θ_p^1, Z_{ϵ_1-1} a linear equation in θ^3_{p-1}, and so on.

It is important now to consider whether the order in which the roots $\sigma_1 \ldots \sigma_{\epsilon_1}$ are taken is of significance in the solution; that is, whether the value of θ^k_{p-1} associated with a particular root σ_k is the same whichever column of the dependent variables this root is associated with, and whether a change in the order necessarily implies a distinct solution, because, if so, the solution would appear to be by no means unique.

The equation X_{ϵ_1+1} giving θ^1_{p-1} is, we have seen, of the form $\phi^1(\sigma_1)\theta^1_{p-1} + \psi(\sigma_1) = 0$, where $\phi^1(\theta) = \dfrac{d}{d\theta}\phi(\theta)$.

Thus
$$\Delta_p X_{\epsilon_1+1} \equiv \frac{\phi^1(\sigma_1)-\phi^1(\sigma_2)}{\sigma_1-\sigma_2}\theta^1_{p-1} + \frac{\psi(\sigma_1)-\psi(\sigma_2)}{\sigma_1-\sigma_2}.$$

Now $\phi^1(\sigma_1)$ is save for a constant factor

$$(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)\ldots(\sigma_1-\sigma_{\epsilon_1}),$$

so that $\Delta_p X_{\epsilon_1+1}$ is

$$\{(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)\ldots(\sigma_1-\sigma_{\epsilon_1}) + (\sigma_2-\sigma_3)\ldots(\sigma_2-\sigma_{\epsilon_1})\}\theta^1_{p-1} + \frac{\psi(\sigma_1)-\psi(\sigma_2)}{\sigma_1-\sigma_2} = 0.$$

But the equation Y_{ϵ_1}, which is independent of $\theta^2_{p-2}, \ldots$ becomes, when $\theta_p^1 (= \sigma_1)$ is substituted for θ_p^2, the equation $\Delta_{p-1}X_{\epsilon_1+2}$, so that when θ_p^1 is substituted for θ_p^2 in Y_{ϵ_1} it becomes symmetrical in θ^1_{p-1} and θ^2_{p-1}. Y_{ϵ_1} must therefore be of the form

$$A(\sigma_1, \sigma_2)\,\theta^1_{p-1}+B(\sigma_1, \sigma_2)\,\theta^2_{p-1}+C(\sigma_1, \sigma_2) = 0,$$

where

$$A(\sigma_1, \sigma_1) = B(\sigma_1, \sigma_1)$$

and

$$A(\sigma_1, \sigma_2)+B(\sigma_1, \sigma_2) = (\sigma_1-\sigma_3)\ldots+(\sigma_2-\sigma_3)\ldots.$$

Hence

$$A(\sigma_1, \sigma_1) = (\sigma_1-\sigma_3)(\sigma_1-\sigma_4)\ldots = B(\sigma_1, \sigma_1).$$

Now before the values of θ_p^1, θ_p^2 are substituted in Y_{ϵ_1} the coefficient of θ^2_{p-1} must be a function of θ_p^2 only.

Therefore B must be a function of σ_2 independent of σ_1.

Hence

$$B = (\sigma_2-\sigma_3)\ldots(\sigma_2-\sigma_{\epsilon_1})$$

and

$$A = (\sigma_1-\sigma_3)\ldots(\sigma_1-\sigma_{\epsilon_1}).$$

The equation for θ^2_{p-1} is therefore

$$(\sigma_2-\sigma_3)\ldots(\sigma_2-\sigma_{\epsilon_1})\,\theta^2_{p-1}+(\sigma_1-\sigma_3)\ldots\theta^1_{p-1}+\frac{\psi(\sigma_1)-\psi(\sigma_2)}{\sigma_1-\sigma_2} = 0.$$

By virtue of the equation giving θ^1_{p-1}, therefore, we have

$$(\sigma_2-\sigma_3)\ldots(\sigma_2-\sigma_{\epsilon_1})\,\theta^2_{p-1}+\frac{\psi(\sigma_2)}{\sigma_2-\sigma_1} = 0$$

as the equation for θ^2_{p-1}.

But this is identically the equation that would have been obtained for θ^1_{p-1} if σ_1, σ_2 had been interchanged. Thus the value of θ_{p-1} associated with the root σ_2 is unaltered by this interchange.

We have further to see that the same permutation does not alter the θ_{p-1} for the subsequent columns.

In just the same way as above the equation for θ^3_{p-1} is shown to be

$$(\sigma_2-\sigma_4)\,(\sigma_2-\sigma_5)\ldots\theta^2_{p-1}+(\sigma_3-\sigma_4)\ldots\theta^3_{p-1}+\frac{\dfrac{\psi(\sigma_2)}{\sigma_2-\sigma_1}-\dfrac{\psi(\sigma_3)}{\sigma_3-\sigma_1}}{\sigma_2-\sigma_3} = 0,$$

which gives

$$(\sigma_3-\sigma_4)\ldots\theta^3_{p-1}+\frac{\psi(\sigma_3)}{(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)} = 0$$

independently of the order of σ_1 and σ_2.

The same holds for θ^k_{p-1}, $k \leqq \epsilon_1$, and a similar proof for the interchange of any other pair of the roots σ.

Thus supposing the roots of $\phi(\theta_p) = 0$ to be all different, there is associated with each a unique determinate value of θ_{p-1}.

These quantities then being determined, consider now the equation X_{ϵ_1+2}, $Y_{\epsilon_1+1}\ldots$ and first it must be pointed out that the relations established between the equations X, Y... through the operators Δ (pp. 14–15), where the quantities θ_p, $\theta_{p-1}\ldots$ were considered as independent, are still valid when θ_{p-1}, &c., are determined as functions of θ_p.

The operator Δ_p in the first place becomes replaced in the equation X_{ϵ_1+2} after that giving $\theta^1{}_{p-1}$ by $\Delta_p+\Delta_{p-1}.\dfrac{\theta^1{}_{p-1}-\theta^2{}_{p-1}}{\theta_p{}^1-\theta_p{}^2}$. But $\Delta_{p-1}X_{\epsilon_1+2}$ vanishes owing to the choice of $\theta^1{}_{p-1}$, so that the value of $\theta^1{}_{p-1}$ being substituted in X_{ϵ_1+2} the operator Δ_p may be still applied to establish a relation with Y_{ϵ_1+1}.

We have further

$$\Delta_{p-2}X_{\epsilon_1+2} = \phi_{\epsilon_1-1}(\theta_p{}^1),$$

while $\Delta_{p-3}X_{\epsilon_1+2}\ldots$ vanish identically because of the vanishing of Y_{ϵ_1-2}, $Y_{\epsilon_1-1}\ldots$.

Thus the equation X_{ϵ_1+2} is of the form

$$\phi_{\epsilon_1-1}(\theta_p{}^1).\theta^1{}_{p-2}+\psi(\theta^1{}_{p-1},\ \theta_p{}^1) = 0,$$

in which $\theta_p{}^1$ and $\theta^1{}_{p-1}$ are to have their determined values, so that the equation may be written

$$\phi_{\epsilon_1-1}(\sigma_1)\theta^1{}_{p-2}+\chi(\sigma_1) = 0.$$

The operation Δ_p having been shown to be applicable to the equation in this form, reasoning exactly as above shows that the equation for $\theta^2{}_{p-2}$ reduces to

$$\phi_{\epsilon_1-1}(\sigma_2)\theta^2{}_{p-2}+\chi(\sigma_2) = 0,$$

so that the values of θ_{p-2} associated with the roots σ_1, σ_2 are independent of the order in which these roots are taken, and likewise the values of $\theta^3{}_{p-2}\ldots$ will be unaltered by a permutation of the same. The same may be shown in the same way of a permutation of any other two consecutive roots, viz., that such permutation gives rise to a corresponding permutation of the $\theta^s{}_{p-2}\ldots$. Identical reasoning leads to an identical conclusion with regard to $\theta_{p-3}\ldots\theta_2$.

Eventually we come to equations giving θ_1. When $\theta_p{}^1\ldots\theta_2{}^1$ have all been found, the equation X_{ϵ_1+p-1} is of the form $\phi_{\epsilon_1-1}(\sigma_1)\theta_1{}^1+\chi(\sigma_1) = 0$, where, as before, the coefficient of $\theta_1{}^1$ is not zero, so that $\theta_1{}^1$ is determined like the rest; while $\theta_1{}^2\ldots\theta_1{}^{\epsilon_1}$ are found respectively from $Y_{\epsilon_1+p-2}\ldots$.

All the θ's being now determined, if we pass to the equation X_{ϵ_1+p} and follow the same argument that was required to prove the preceding equations independent of $x_1{}^1\ldots$, the coefficient of $x_1{}^1$ is found to be the left-hand member of X_{ϵ_1+p-1} with $(\theta_1{}^1+1)$ substituted for $\theta_1{}^1$, *i.e.*, it is simply $\phi_{\epsilon_1-1}(\sigma_1)$, which is not zero. Thus X_{ϵ_1+p} determines the first of the undetermined elements $x_1{}^1\ldots$.

Similarly in X_{ϵ_1+p+1} the coefficient of $x_2{}^1$ is $2\phi_{\epsilon_1-1}(\sigma_1)$, so that by this $x_2{}^1$ is given, and so on for the succeeding equations in turn.

In order to proceed to the determination of the second column it may be noticed at once that the coefficient of y_r^1 in any equation is exactly equal to that of x_r^1 in the corresponding equation from the first column with σ_1 and σ_2 interchanged, which includes the interchange of θ^1_{p-k} and θ^2_{p-k}. In the equation Y_{ϵ_1+p-1}, therefore, the coefficient of y_1^1 is identically zero, while the unknown θ_2's are now all determined. The closer consideration of this equation is deferred for a moment.

The coefficient of y_1^1 in Y_{ϵ_1+p} = coefficient of x_1^1 in X_{ϵ_1+p} with σ_1 and σ_2 interchanged

$$= (\sigma_2-\sigma_1)(\sigma_2-\sigma_3)\ldots(\sigma_2-\sigma_{\epsilon_1}) \neq 0.$$

Similarly the coefficient of y_2^1 in Y_{ϵ_1+p+1}, and in general of y_k^1 in $Y_{\epsilon_1+p+k-1}$, is not zero, while $Y_{\epsilon_1+p+k-1}$ does not contain any element y_m^1 for which $m>k$.

For the third column the equation Z_{ϵ_1+p-3} determines θ_1^3, and the two equations following, Z_{ϵ_1+p-2}, Z_{ϵ_1+p-1}, are still independent of z_1^1, $z_2^1\ldots$, while the equation $Z_{\epsilon_1+p+k-1}$ contains $z_1^1\ldots z_k^1$, the coefficient of z_k^1 being $k(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)(\sigma_3-\sigma_4)\ldots$.

In the same way, if the elements of the ϵ_1^{th} column be denoted by ξ and the associated equations by Ω, the equation Ω_p determines $\theta_1^{\epsilon_1}$, $\Omega_{p+1}\ldots\Omega_{p+\epsilon_1-1}$ do not contain $\xi_1^1\ldots$, and $\Omega_{p+\epsilon_1}$ contains ξ_1^1 only.

11. So far the matrices $\chi_p\ldots\chi_1$ have been taken to be simply diagonals. It will now be shown that the insertion of constants to the right of the diagonal in the first ϵ_1 columns of χ_p can be carried out in such a manner as to affect none of the conclusions hitherto made, while they may be chosen so that the equations Y_{ϵ_1+p-1}, Z_{ϵ_1+p-2}, Z_{ϵ_1+p-1}, &c., are all satisfied.

Denoting by α_{ij} a constant in the i^{th} column and j^{th} row, $i>j$, $i \geqq \epsilon_1$, $j<\epsilon_1$, the Y equations become

$$(0) \quad (\alpha_{p+1}-\theta_{p+1})y_0-x_0=0.$$

$$(1) \quad (\alpha_{p+1}-\theta_{p+1})y_1+(\alpha_p-\theta_p)y_0-x_1-\alpha^{21}x_0=0,$$

$$\cdots\cdots\cdots\cdots$$

$$(r+1) \quad (\alpha_{p+1}-\theta_{p+1})y_{r+1}+(\alpha_p-\theta_p)y_r\ldots-x_{r+1}-\alpha^{21}x_r=0.$$

These equations are to be treated just as they were before the constants α were introduced—the same elements remain undetermined as before, but at each stage the quantities found presumably contain α^{21}.

We see at once from equation (1) that the coefficient of α_{21} in y_1 (the first element being excepted) is simply y_0. In fact, it can be shown step by step that the coefficient of α_{21} in y_{r+1} (the first element always excepted) is exactly that part of y_r which is independent of α_{21} with θ_1^2 increased by 1; and therefore the coefficient of α^{21} in Y_{r+1}, when the values of $y_r^2\ldots$ as far as they are known are substituted, is equal to that part of the left-hand member of Y_r which is independent of α^{21}, θ_1^2 being increased by unity.

Now Y_1 is independent of α^{21} and of θ_1^2 and vanishes, thus Y_2 is independent of α^{21}, and is therefore the same as if α^{21} were zero. It is also independent of θ_1^2, so that Y_3 again is independent of α^{21}.

Thus until k is so large that Y_k does not vanish independently of θ_1^2, Y_{k+1} is independent of α^{21}, and therefore the same as was obtained in the foregoing, where α^{21} was neglected.

Thus the insertion of α^{21} in χ_p does not affect any of the equations $Y_1 \ldots Y_{\epsilon_1+p-2}$, and therefore the values of $\theta_p^2 \ldots \theta_1^2$ are independent of α^{21}.

But in the equation Y_{ϵ_1+p-1} the coefficient of α^{21} is the left-hand member of Y_{ϵ_1+p-2} with θ_1^2+1 for θ_1^2

$$= (\sigma_2-\sigma_3)(\sigma_2-\sigma_4)\ldots \neq 0.$$

Thus α^{21} can certainly be chosen so that the equation Y_{ϵ_1+p-1} is satisfied.

Having determined α^{21}, it is at once seen from p. 19 that the following equations now determine y_1^1, $y_2^1 \ldots$ without ambiguity; for since $\theta_p^2 \ldots \theta_1^2$ are independent of α^{21}, the coefficients of y_1^1, &c., are those found there whether α^{21} be zero or not.

In the same way for the third column, with α^{32}, α^{31}, taken into account, the equations become

$$(\alpha_{p+1}-\theta^3{}_{p+1})\,z_0 - y_0 = 0,$$

$$(\alpha_{p+1}-\theta^3{}_{p+1})\,z_1 + (\alpha_p-\theta_p{}^3)\,z_0 - y_1 - \alpha_{32}y_0 - \alpha_{31}x_0 = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and just as before the first equation in which α_{32} occurs with a non-vanishing coefficient is the one following the equation from which θ_1^3 first does not vanish out identically, viz., Z_{ϵ_1+p-2}; while α_{31} will occur first in the equation homologous to the Y equation in which α_{21} first occurred, viz., Z_{ϵ_1+p-1}; in fact, in Z_{ϵ_1+p-2}, α_{32} will occur multiplied by the left-hand member of Z_{ϵ_1+p-3} with θ_1^3+1 put for θ_1^3, and in Z_{ϵ_1+p-2}, α_{31} will be multiplied by the left-hand member of Y_{ϵ_1+p-2} with θ_1^3+1 put for θ_1^2: both these factors are other than zero.

Thus α^{32} can be chosen to satisfy Z_{ϵ_1+p-3}, and subsequently α^{31} to satisfy Z_{ϵ_1+p-2}, while the preceding equations are quite independent of them both; just as for y, then, $z_1^1 \ldots$ are given in succession without ambiguity.

Treating the remainder of the first ϵ_1 columns in just the same way, all the elements of these columns are found in succession, and the solution is complete as far as these columns are concerned.

The ϵ_2 columns associated with the next group of equal roots may be treated in the same way, the singular equations being in this case the $(\epsilon_1+\epsilon_2)^{\text{th}}$ of each set; constants α^{ij} will again be chosen in the matrix χ to the right of the diagonal, $\epsilon_1+\epsilon_2 \geqq \iota > \epsilon_1+1$, $\epsilon_1+\epsilon_2 > j \geqq \epsilon_1+1$, to satisfy certain equations as above, and so for each root in succession.

Thus if the various equations for θ_p associated with the different groups of equal roots of the characteristic equation have their roots all different, and the "equations of condition" (p. 15) for each root are satisfied, a formal solution of the linear system has been found in the form

$$\eta\Omega\left(\frac{\alpha_{p+1}}{t^{p+1}}+\frac{\chi_p}{t^p}+\ldots+\frac{\chi_1}{t}\right),$$

where the elements of η are series of positive integral ascending powers of t, and $\chi_1\ldots\chi_{p-1}$ have all elements zero save those in the diagonals, which are made up of determinate constants; and χ_p consists merely of square matrices about its diagonal of $\epsilon_1, \epsilon_2\ldots$ rows and columns respectively, each of which has zero everywhere to the left of the diagonal and determinate constants everywhere else. The elements of η are in general divergent.

The matrix Ω above will be known as the "determining matrizant." As occasion will be found later to discuss a more general matrizant, nothing further will be said of it here except for the case in which $p = 1$, which will be worked out fully in order to make clear the march of ideas in the more general case.

12. For $p = 1$ the equation X_{ϵ_1} is an equation of degree ϵ_1 for θ_1^1, Y_{ϵ_1-1} is of degree ϵ_1-1 for θ_1^2, and so on.

If

$$X_{\epsilon_1} \equiv \phi(\theta_1^1) = 0, \text{ then } Y_{\epsilon_1-1} \equiv \frac{\phi(\theta_1^1)-\phi(\theta_1^2-1)}{\theta_1^1-\theta_1^2+1},$$

so that the remaining roots are those of Y_{ϵ_1-1} diminished by unity.

Similarly the roots of Y_{ϵ_1-1} are those of Z_{ϵ_1-2} diminished by unity, and so on, so that the roots of $\phi(\theta_1^1) = 0$ are $\theta_1^1, \theta_1^2-1, \ldots, \theta_1^{\epsilon_1}-\epsilon_1-1$.

The equation

$$X_{\epsilon_1+1} \text{ is } x_1^1\phi(\theta_1^1+1)+\chi(\theta_1^1) = 0;$$

$$X_{\epsilon_1+2} \text{ is } x_2^1\phi(\theta_1^1+2)+x_1^1\psi(\theta_1^1)+\chi^1(\theta_1^1) = 0,$$

and so on. The equations for $y_1^1\ldots$ are of the same form, with θ_1^2 for θ_1^1. We suppose therefore in the first place that $\theta_1^1, \theta_1^2\ldots$ do not differ by positive integers or by zero, so that the coefficients of the first terms in these equations are all other than zero, and all the x's and y's are determined uniquely. The quantities α being then determined, as above, the solution is altogether determinate.

If $p = 1$ and $\theta_1^1\ldots\theta_1^{\epsilon_1}$ do not differ among themselves by integers, then the solution is of the form

$$\eta\Omega\left\{\begin{pmatrix}\theta_2^1, & 1 & 0 & \ldots\\ & \theta_2^2 & 1 & 0\ldots\\ . & . & . & .\end{pmatrix}\frac{1}{t^2}+\begin{pmatrix}\theta_1^1 & \alpha^{21} & \alpha^{31}\ldots\\ & \theta_1^2 & \alpha^{32}\ldots\\ . & . & .\end{pmatrix}\frac{1}{t}\right\}$$

$$=\eta\Omega\left\{\begin{pmatrix}\theta_2^1 & 0 & \ldots\\ & \theta_2^2 & 0\ldots\\ . & . & .\end{pmatrix}\frac{1}{t^2}+\begin{pmatrix}\theta_1^1 & 0 & \ldots\\ & \theta_1^2 & 0\ldots\\ . & . & .\end{pmatrix}\frac{1}{t}+\begin{pmatrix}0 & 1 & \ldots\\ 0 & 0 & 1\ldots\\ . & . & .\end{pmatrix}\frac{1}{t^2}+\begin{pmatrix}0 & \alpha^{21} & \ldots\\ & 0 & \alpha^{31}\ldots\\ . & . & .\end{pmatrix}\frac{1}{t}\right\}.$$

in which $\alpha^{ij} = 0$, unless $\theta_2^i = \theta_2^j$

$$= r\Omega\left\{\frac{\theta_2}{t^2}+\frac{\theta_1}{t}+\frac{\gamma_2}{t^2}+\frac{\gamma_1}{t}\right\}, \text{ say.}$$

Now $\Omega(w+\sigma) = \Omega(w)\,\Omega\{\Omega^{-1}(w)\,\sigma\Omega(w)\}$ (Dr. BAKER, *loc. cit.*, p. 339), and

$$\Omega\left(\frac{\theta_2}{t^2}+\frac{\theta_1}{t}\right) = \omega\omega_0^{-1}, \text{ where } \omega = \begin{pmatrix} e^{-\frac{\theta_2^1}{t}}t^{\theta_1^1} & 0 & \ldots \\ & e^{-\frac{\theta_2^2}{t}}t^{\theta_1^2} & \ldots \end{pmatrix}$$

and ω_0 is the value of ω at $t = t_0^0$, so that

$$\Omega\left(\frac{\theta_2}{t^2}+\frac{\theta_1}{t}+\frac{\gamma_2}{t^2}+\frac{\gamma_1}{t}\right)$$

$$= \frac{\omega}{\omega_0}\,\Omega\left\{\frac{1}{t^2}\begin{pmatrix} 0, & \left(\frac{t}{t_0}\right)^{-\theta_1^1+\theta_1^2}, & 0 & ,\ldots \\ 0, & 0 & , \left(\frac{t}{t_0}\right)^{-\theta_1^2+\theta_1^3} & ,\ldots \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} + \frac{1}{t}\begin{pmatrix} 0, & \alpha^{21}\left(\frac{t}{t_0}\right)^{-\theta_1^1+\theta_1^2}, & \alpha^{31}\left(\frac{t}{t_0}\right)^{-\theta_1^1+\theta_1^3} & \ldots \\ 0, & 0 & , \alpha^{32}\left(\frac{t}{t_0}\right)^{-\theta_1^2+\theta_1^3} & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}\right\},$$

there being no exponentials in the last matrix since

$$\alpha^{ij} = 0, \text{ unless } \theta_2^i = \theta_2^j.$$

Writing this $\eta\Omega\left(\frac{\theta_2}{t^2}+\frac{\theta_1}{t}\right)\Omega\left(\frac{\Gamma}{t^2}\right)$, Γ is a matrix having zero in and to the left of the diagonal, so that $Q\left(\frac{\Gamma}{t^2}\right)$ has zeros in the same places.

$\frac{\Gamma}{t^2}Q\left(\frac{\Gamma}{t^2}\right)$ therefore has zeros in and to the left of the diagonal, and also in the $(n-1)$ places to the right of the diagonal, and also wherever Γ has a zero, and so on.

Thus $Q\frac{\Gamma}{t^2}Q\frac{\Gamma}{t^2}\ldots$ vanishes after a finite number of steps. Further, none of these expressions contain $\log t$, since Γ contains no integral powers of t.

Thus

$$\Omega\left(\frac{\Gamma}{t^2}\right) = \begin{pmatrix} 1, & \gamma_{21}, & \gamma_{31}\ldots \\ & 1, & \gamma_{32} \\ & & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix};$$

all the places which were occupied by zeros in Γ being also occupied by zeros in this, and γ_{ij} contains only a finite number of powers of t, positive or negative, and no logarithms.

We may specify a little more exactly the form of the term γ_{ij}. A typical term of Γ/t^2 is

$$\frac{c_{ij}}{t^2}\left(\frac{t}{t_0}\right)^{\theta_j-\theta_i}+\frac{\alpha_{ij}}{t}\left(\frac{t}{t_0}\right)^{\theta_j-\theta_i},$$

and $\theta_j-\theta_i$ is not an integer and c_{ij} is unity or zero.

The corresponding term in $Q(\Gamma/t^2)$ is

$$C_{ij}\left\{\left(\frac{t}{t_0}\right)^{\theta_j-\theta_i-1}-1\right\}+A_{ij}\left\{\left(\frac{t}{t_0}\right)^{\theta_j-\theta_i}-1\right\}.$$

It follows that in $\frac{\Gamma}{t^2}Q\frac{\Gamma}{t^2}$ the r^{th} column is a sum of terms belonging to the indices $\theta_1^s-\theta_1^i$, where $s=1, 2\ldots r$, and so for each operation Q.

Thus finally we have the result—

The term in the i^{th} row and j^{th} column of $\Omega\left(\frac{\Gamma}{t^2}\right)$ is a sum of terms belonging to the indices $\theta_1^s-\theta_1^i$, $s=1, 2\ldots j$; so that in the product $\Omega\left(\frac{\theta_2}{t^2}+\frac{\theta_1}{t}\right)$ the j^{th} column is a sum of terms belonging to the indices $\theta_1^1\ldots\theta_1^j$.

13. Supposing still that $p=1$, let the indices $\theta_1^1\ldots\theta_1^{\epsilon_1}$ cease to differ by other than integers, and likewise the other groups. Let them be arranged in groups differing by integers, so that their real parts are in descending order of magnitude in each group.

Then no root of $\phi(\theta_1^1)=0$ will exceed θ_1^1 by an integer, and therefore $\phi(\theta_1^1+k)\neq 0-k$ a positive integer, so that the equations X_{ϵ_1+1}, … do not fail to determine x_1^1….

If, however, $\theta_1^2=\theta_1^1-k$, $\phi(\theta_1^2+k)=0$; so that the coefficient of y_k^1 in Y_{ϵ_1+k} vanishes, leaving y_k^1 unknown. We take $y_k^1=0$ as the simplest assumption, and the following equations then give y^1_{k+1}, &c., all without ambiguity. We are, however, left with Y_{ϵ_1} and Y_{ϵ_1+k} in general unsatisfied. Of these one can in general be satisfied without affecting the rest of the argument by an adjustment of the element χ_p^{21}.

It has been seen that a constant α^{21} in the matrix χ_1 occurs first with non-vanishing coefficient in Y_{ϵ_1}.

Clearly, then, if we introduce $\alpha^{21}t^k$, it will leave all the equations to Y_{ϵ_1+k-1} unaffected, and add to Y_{ϵ_1+k} the left-hand member of Y_{ϵ_1-1} with θ_1^2+k+1 for θ_1^2.

But Y_{ϵ_1-1} is an equation of degree ϵ_1-1 of which θ_1^2 is the greatest root, so that the multiple of α^{21} added to Y_{ϵ_1+k} is not zero. Thus a proper choice of α^{21} satisfies Y_{ϵ_1+k}.

Again suppose $\theta_1^3=\theta_1^2-k_1=\theta_1^1-k_1-k_2$, $k_1>0$, $k_2>0$.

Then the equations Z_{ϵ_1+k}, $Z_{\epsilon_1+k_1+k_2}$ fail to give $z^1{}_{k_1}$, $z^1{}_{k_1+k_2}$; but α^{31}, α^{21} can again be determined so that, if $\alpha^{31}t^{k_1+k_2}$, $\alpha^{32}t^{k_1}$ occupy the places above θ_1^3 in χ_1, the equations $Z_{\epsilon_1+k_1}$, $Z_{\epsilon_1+k_1+k_2}$ are satisfied, θ_1^3 being unaffected and $z^1{}_{k_1}$, $z^1{}_{k_1+k_2}$ being taken zero.

Suppose then $\theta_1^1\ldots\theta_1^h$ form the first group of $\theta_1^1\ldots\theta_1^{\epsilon_1}$ differing by integers. Then treating the first h columns all in the same way, the $\frac{1}{2}h(h-1)$ equations Y_{ϵ_1}, Z_{ϵ_1}, Z_{ϵ_1-1}… must be satisfied identically when the θ_1^s have been all determined, and must be added to the equations of condition already found.

Suppose now $\theta_1^{h+1}\ldots\theta_1^{h+k}$ form the next group of roots differing by integers and consider the $(h+r)^{\text{th}}$ column $r<k$. The equation giving θ_1^{h+r} is that indicated by the suffix $\epsilon_1-(h+r-1)$, and those following this up to that with the suffix ϵ_1 are independent of the undetermined elements of η.

Further $r-1$ of the equations subsequent to these fail to determine the appropriate element as above, on account of the quantities $\theta_1^{h+s}-\theta_1^{h+r}$ being positive integers for $s = 1, 2\ldots(r-1)$. These $(r-1)$ equations are satisfied by putting terms $t^{\theta_1^{h+k}-\theta_1^{h+r}}\alpha^{h+r,h+k}(k = 1\ldots r-1)$ in χ_1, while of the other $h+r-1$ equations constants $\alpha^{h+r,s}(s = 1\ldots h)$ can be found to satisfy h. Thus $(r-1)$ equations of condition are found from this column, and therefore $\frac{1}{2}k(k-1)$ from this group of roots; and so for each group of roots.

Assuming all the equations of condition to be satisfied, we have now the following formal solution

$$\eta\Omega\left[\frac{\alpha_2}{t^2}+\frac{\chi_1}{t}\right],$$

where χ_1 is as follows:—

The square matrices about the diagonal of $h, k\ldots$ rows and columns respectively, corresponding to the groups of $\theta_1^1\ldots$ which differ by integers, are of the form

$$\begin{pmatrix} \theta_1^1 & \alpha_{21}t^{\theta_1^1-\theta_1^2}\ldots & \alpha_{h_1}t^{\theta_1^1-\theta_1^h} \\ & \theta_1^2 & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ & & \alpha_{h,h-1}t^{\theta_1^{h-1}-\theta_1^h} \\ & & \theta_1^h \end{pmatrix};$$

and all other elements, to the right of the diagonal and within the matrices of $\epsilon_1\ldots$ rows and columns about the diagonal corresponding to the groups of equal roots for θ_2, are numerical constants, and all others to the left and right of the diagonal are zero.

Applying now the formula

$$\Omega(w+\sigma) = \Omega(w)\Omega\{\Omega^{-1}(w)\sigma\Omega(w)\},$$

the solution is put in the form

$$\eta\Omega\left(\frac{\theta_2}{t^2}+\frac{\theta_1}{t}\right)\Omega_1\left\{\frac{1}{t^2}\begin{pmatrix} 0, & (t/t_0)^{-\theta_1^1+\theta_1^2},\ldots & \\ 0, & 0, & (t/t_0)^{-\theta_1^1+\theta_1^2},\ldots \\ \cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \end{pmatrix}+\frac{1}{t}\begin{pmatrix} 0 & c_{21} & c_{31}\ldots \\ 0 & 0 & c_{32}\ldots \\ \cdot & \cdot & \cdot\quad\cdot \end{pmatrix}\right\},$$

where in the last matrix all elements are zero that were zero in χ_1, and c_{ij} is a constant if $\theta_j-\theta_i$ is a positive integer, but otherwise is a numerical multiple of $t^{\theta_j-\theta_i}$.

The expansion of the matrizant can be effected as on p. 22, with the result that in the expanded matrix the first h elements of the first row contain $\log(t/t_0)$ to the powers $0, 1\ldots h-1$ respectively, while the rest of the row is free from logarithms; the second row begins with zero, then unity, and the next $(h-2)$ elements contain $\log(t/t_0)$ to powers $1\ldots(h-2)$ respectively, and so on, the h^{th} row being entirely free from logarithms. In the $(h+1)^{\text{th}}$ row in the k places beginning with the diagonal term $\log(t/t_0)$ occurs to powers $0, 1\ldots(k-1)$, and so on.

14. Returning now to the general case left out of consideration on p. 16, in which the roots of $\phi(\theta_p) = 0$ are not all unequal, we suppose the roots of this equation arranged in groups, of which the members are all equal.

If $\sigma_1 = \sigma_2 = \ldots = \sigma_r$, $\phi_{\epsilon_1-1}(\sigma_1) = \phi_{\epsilon_1-1}(\sigma_2) = 0$, since the roots of $\phi_{\epsilon_1-1}(\theta) = 0$ are $\sigma_2, \ldots \sigma_r$; and again, $\phi_{\epsilon_1-2}(\sigma_2)$, $\phi_{\epsilon_1-3}(\sigma_3) \ldots \phi_{(\epsilon_1-r+1)}(\sigma_{r-1})$ are all zero, where $\phi_{\epsilon_1-2} = 0 \ldots \phi_{\epsilon_1-r+1} = 0$ are the equations for $\theta_p^{\,3}, \theta_p^{\,4} \ldots \theta_p^{\,r}$.

The equation for $\theta^1_{\ p-1}$ (p. 16) reduces therefore simply to $\psi(\sigma_1) = 0$ and fails to determine $\theta^1_{\ p-1}$; but, σ_1 being already known, this must be merely an "equation of condition" among the coefficients.

Similarly in the equation X_{ϵ_1+2} the coefficient of $\theta^1_{\ p-2}$ vanishes, and this equation therefore is of the form

$$\psi(\theta^1_{\ p-1}, \theta_p^{\,1}) = 0 \quad \text{or} \quad \psi(\theta^1_{\ p-1}, \sigma_1) = 0.$$

Now the operator Δ_{p-1} acting on this equation, since $\sigma_1 = \sigma_2$, gives the equation Y_{ϵ_1}, which, as has been seen (p. 17), is linear in $\theta^2_{\ p-1}$, the coefficient being $\phi_{\epsilon_1-2}(\sigma_2)$.

If $r = 2$, this does not vanish, and therefore X_{ϵ_1+2} is a quadratic for $\theta^1_{\ p-1}$, of which, owing to the relation through Δ_{p-1}, $\theta^1_{\ p-1}$, $\theta^2_{\ p-1}$ are the two roots.

If, however, $r > 2$, the equation Y_{ϵ_1} must become an equation of condition, since $\phi_{\epsilon_1-2}(\sigma_2) = 0$, and therefore also X_{ϵ_1+2} becomes independent of $\theta^1_{\ p-1}$ and gives another equation of condition.

Carrying on this reasoning step by step, we find that $X_{\epsilon_1+1} \ldots X_{\epsilon_1+r-1}$ are all independent of $\theta^1_{\ p-1}, \theta^1_{\ p-2} \ldots$, while X_{ϵ_1+r} is of degree r in $\theta^1_{\ p-1}$ and independent of $\theta^1_{\ p-2} \ldots$. If any root of this equation be taken for $\theta^1_{\ p-1}$ the equations $Y_{\epsilon_1} \ldots Y_{\epsilon_1+r-3}$ are independent of $\theta^2_{\ p-1}, \theta^2_{\ p-2} \ldots$, while Y_{ϵ_1+r-2} is an equation of degree $r-1$, which, since it is derived from X_{ϵ_1+r} by the operator Δ_{p-1}, has for its roots the remaining roots of X_{ϵ_1+r}.

Choosing one of these for $\theta^2_{\ p-1}$, Z_{ϵ_1+r-4} gives an equation of degree $r-2$ whose roots are the remaining, and so on.

Similarly, if $\sigma_{r+1} = \ldots = \sigma_{r+s}$, $\theta_{p-1}^{\ r+1} \ldots \theta_{p-1}^{\ r+s}$ are given as the roots of an equation of degree s, and so for each group of equal roots σ.

Consider now one such group with the values of $\theta_{p-1}^{\ r+1}, \ldots \theta_{p-1}^{\ r+s}$ obtained.

Let the equations of which these are the roots be

$$\phi_s(\theta_{p-1}^{\ r+1}) = 0, \quad \phi_{s-1}(\theta_{p-1}^{\ r+2}) = 0, \quad \ldots, \quad \phi_1(\theta_{p-1}^{\ r+s}) = 0.$$

Then, if the roots of ϕ_s be all unequal, say $= \tau_1 \ldots \tau_s$,

$$\phi_{s-1}(\tau_1) \neq 0, \quad \phi_{s-2}(\tau_2) \neq 0, \quad \ldots,$$

but

$$\phi_{s-1}(\tau_j) = 0, \; j \neq 1.$$

The subsequent equations are then seen by the application of Δ_3 to be

$$\phi_{s-1}(\tau_1)\theta_{p-2}{}^{r+1}+\chi_s(\tau_1, \sigma_1) = 0,$$

$$\phi_{s-2}(\tau_2)\theta_{p-2}{}^{r+2}+\chi_{s-1}(\tau_2, \sigma_2, \tau_1, \sigma_1) = 0,$$

. . . .

which, since the coefficients of θ_{p-2} do not vanish, at once give the values of $\theta_{p-2}{}^{r+1}\ldots$; these, as in the case of θ_{p-1} when the σ's were unequal, can be shown, if the roots $\tau_1\ldots$ undergo a permutation, to undergo the same permutation, so that the same θ_{p-2} is associated with any particular τ in whatever place this τ is taken.

If, however, the roots τ fall into groups of which the members are equal to one another, these equations again resolve themselves into equations of condition owing to the vanishing of $\phi_{s-1}(\tau_1)$, &c.; and, as before, the quantities θ_{p-2} fall into corresponding groups given as the roots of equations of degrees equal to the numbers in the respective groups.

The process may clearly be carried on as far as the determination of θ_2 by the use of the operators $\Delta_{p-2}\ldots\Delta_2$.

A further remark should be made as to the finding of θ_1, in connection with the operator Δ_1, which has been defined to be such that

$$\Delta_1 X(\theta_1^1) = \frac{-X(\theta_1^2-p)+X(\theta_1^1)}{\theta_1^1-\theta_1^2+p}.$$

Suppose that $\theta_2^1\ldots\theta_2^k$ are given by a set of equations

$$\omega_k(\theta_2^1) = 0, \quad \omega_{k-1}(\theta_2^2) = 0, \quad \ldots, \quad \omega_1(\theta_2^k) = 0,$$

where the affixes of the ω's denote the degrees of the equations, and the roots of each equation are the remaining roots of the preceding after any one of them has been chosen.

Suppose that of these $\theta_2^1\ldots\theta_2^h$ are equal, so that

$$\theta_r^1 = \theta_r^2\ldots = \theta_r^h, \quad r = p+1, p, \ldots 2.$$

Then $\omega_{k-1}(\theta_2^1) = 0$, $\omega_{k-2}(\theta_2^2) = 0$, $\ldots$, $\omega_{k-h+1}(\theta_2^{h-1}) = 0$, but $\omega_{k-h}(\theta_2^h) \neq 0$.

Then if the X equations following ω_k be denoted successively by

$${}_1\omega_k = 0, \quad {}_2\omega_k = 0, \quad \ldots,$$

${}_1\omega_k$ is independent of x_1^1, &c., by virtue of $\omega_k = 0$ and the preceding equations, and therefore

$$\Delta_1({}_1\omega_k) \equiv \omega_{k-1}(\theta_2^1) = 0, \text{ since } \theta_p^1 = \theta_p^2, \ \theta^1_{p-1} = \theta^2_{p-1}, \ \ldots,$$

so that ${}_1\omega_k$ is also independent of θ_1^1 and must therefore vanish identically when θ_2^1 is determined. Hence also ${}_2\omega_k$ is independent of x_1^1, &c., and therefore

$$\Delta_p({}_2\omega_k) \equiv {}_1\omega_{k-1}(\theta_2^1).$$

But in the same way

$$\Delta_p({}_1\omega_{k-1}) \equiv \omega_{k-2}(\theta_2^{\,2}) = 0,$$

so that ${}_1\omega_{k-1}(\theta_2^{\,2})$ must also vanish when $\theta_2^{\,2}$ is found and thus ${}_1\omega_{k-1}(\theta_2^{\,1}) = 0$, so that ${}_2\omega_k$ is also independent of $\theta_1^{\,1}$.

Ultimately we have

$$\begin{aligned}\omega_{k-h}(\theta_2^{\,h}) &= \Delta_1\{{}_1\omega_{k-h+1}(\theta_2^{\,h})\},\\ &= \Delta_1\Delta_1\{{}_2\omega_{k-h+2}(\theta_2^{\,h})\} = \ldots,\\ &= (\Delta_1)^h\{{}_h\omega_k(\theta_2^{\,h})\},\end{aligned}$$

and $\omega_{k-h}(\theta_2^{\,h}) \neq 0$ and is independent of $\theta_1^{\,h}$.

Thus ${}_h\omega_k(\theta_1^{\,1})$ is an equation of degree h for $\theta_1^{\,1}$, since $\theta_2^{\,1} = \theta_2^{\,h}$, $\theta_3^{\,1} = \theta_3^{\,h}\ldots$.

Suppose its roots are $v_1, v_2, \ldots v_h$.

Take $\theta_1^{\,1}$ to be v_1. Then ${}_{h-1}\omega_{k-1}$ is an equation of degree $h-1$ for $\theta_1^{\,2}$, and its roots are $(p+v_2)(p+v_3)\ldots$, as shown by effecting the operation Δ_1.

Choosing one of these again to be $\theta_1^{\,2}$, say $p+v_2$, ${}_{h-2}\omega_{k-2}$ is an equation for $\theta_1^{\,3}$, whose roots are $(2p+v_3)$, $(2p+v_4)\ldots$. Thus the quantities $\theta_1^{\,1}\ldots\theta_1^{\,h}$ are v_1, $(p+v_2)$, $(2p+v_3)\ldots$, $(h-1)p+v_h$.

In order to particularise the order of the roots $v_1, v_2, \ldots$, they are arranged as soon as found as follows:—

Let all those roots which differ from one another by integers be grouped consecutively and let the arrangement in each group be such that $\theta^{r-1}-\theta^r = 0$ or a positive integer. Suppose, now, the equation ${}_h\omega_k(\theta_1^{\,1}) = 0$ is the equation X_q.

Then the coefficient of $x_1^{\,1}$ in X_{q+1} is ${}_h\omega_k(\theta_1^{\,1}+1)$ which, since no root of ${}_h\omega_k(\theta) = 0$ exceeds $\theta_1^{\,1}$ by a positive integer, does not vanish. X_{q+1} is moreover independent of $x_2^{\,1}, \ldots$, owing to the equations ${}_{h-1}\omega_k = 0, \ldots$, being satisfied independently of $\theta_1^{\,1}$.

Thus X_{q+1} determines $x_1^{\,1}$.

Similarly, X_{q+2} gives $x_2^{\,1}$, the coefficient being ${}_h\omega_k(\theta_1^{\,1}+2)$, and so forth.

Of the Y equations, that determining $\theta_1^{\,2}$ is obtained from X_q by the operator Δ_1. It is, in fact, Y_{q-p}.

The coefficient of $y_1^{\,1}$ in Y_{q-p+1} is equal to the coefficient of $x_1^{\,1}$ in X_{q-p+1} with $\theta_1^{\,2}$, $\theta_2^{\,2}, \ldots$, substituted for $\theta_1^{\,1}$, $\theta_2^{\,1}, \ldots$. But X_{q-p+1} vanishes identically as far as $\theta_1^{\,1}$ is concerned and $\theta_2^{\,2}, \ldots$, are the same as $\theta_1^{\,2}, \ldots$. Thus Y_{q-p+1} is independent of $y_1^{\,1}$. Similarly, $Y_{q-p+2}\ldots Y_{q-1}$ are all independent of the undetermined elements of y.

Suppose now $\theta_1^{\,1}-\theta_1^{\,2} = \lambda$ (a positive integer).

Then ${}_h\omega_k(\theta_1^{\,2}+\lambda) = 0$.

The equation Y_{q+k} contains $y_1^{\,1}, \ldots, y_k^{\,1}$, the coefficient of the last of these being ${}_h\omega_k(\theta_1^{\,2}+k)$ which vanishes for $k = \lambda$.

If, now, in the matrix $\chi_r (r = p, p-1\ldots 2)$ the second element of the first row be $c_r^{\,21}$, and in χ_1 be $c^{21}t^\lambda$, the constants c will, as before $(p = 1)$, affect first the equations $Y_{q-p+1}\ldots Y_{q-1}$ and $Y_{q+\lambda}$ respectively, entering into these with non-vanishing coefficients. Let $c_r^{\,21}$ be determined then to satisfy the first $p-1$ of these equations; $Y_{q+1}\ldots Y_{q+\lambda-1}$

then give $y_1^1 \ldots y^1_{\lambda-1}$. $Y_{q+\lambda}$ then fails to give y_λ^1, but c^{21} can be chosen to satisfy the equation and y_λ^1 may be taken zero.

The following equations then give the remaining elements in succession.

This leaves the equation Y_q in general unsatisfied, and a further equation of condition is therefore necessary.

Similarly, if $\theta_1^2 - \theta_1^3 = \mu$ (an integer) of the equations Z, we can, by proper choice of c_r^{32} $(r = p \ldots 2)$, satisfy the $p-1$ following that which determines θ_1^3, viz., Z_{q-2p}; and just as a proper choice of the constants c_r^{21} enabled us to satisfy $Y_{q-p+1} \ldots Y_{q-1}$, the constants c_r^{31} can be chosen to satisfy $Z_{q-p+1} \ldots Z_{q-1}$.

Thus two equations, Z_{q-p}, Z_q, are left unsatisfied in general. The two remaining constants, c_1^{32} and c_1^{31}, are utilised to satisfy the equations $Z_{q+\mu}$, $Z_{q+\lambda+\mu}$, in which the coefficients of z_μ^1 and $z^1_{\lambda+\mu}$ vanish respectively. To do this the terms $c^{31}t^{\lambda+\mu}$ and $c^{32}t^{\mu+p}$ are inserted in the third column of χ_1.

If then the indices $\theta_1^1 \ldots \theta_1^l$ be equal, or differ from one another by integers, exactly similar treatment applies for each of the first l columns of η, the i^{th} column furnishing $(i-1)$ equations of condition.

For the $(l+1)^{\text{th}}$ column, however, $\theta_1^r - \theta_1^{l+1}$, $(r \gtrless h)$ is not equal to zero or a positive integer. Thus ${}_h\omega_k\,(\theta_1^{l+1}+m)$ does not vanish for any value of m, and the lp constants $c_r^{l+1,s}$ $(r = p \ldots 1,\ s = l \ldots 1)$ can be determined to satisfy the lp equations between U_{q-lp} and U_{q+1}, U denoting an equation of the $l+1^{\text{th}}$ column, and, in particular, U_{q-lp} being the equation determining θ_1^{l+1} and U_{q+1} determining u_1^1.

For the next column, however, $\theta_1^{l+1} - \theta_1^{l+2}$ may be a positive integer, λ^1 say, so that ${}_h\omega_k\,(\theta_1^{h+2}+\lambda^1) = 0$, and the $(q+\lambda^1)^{\text{th}}$ equation, instead of determining the appropriate element of η, can only be satisfied at the expense of the q^{th}, by making the element above θ_1^{l+2} in $\chi_1 - c^{l+2,l+1}t^{\lambda^1}$. The q^{th} equation then becomes a further equation of condition. Thus we shall obtain $r-1$ equations of condition from the $(l+r)^{\text{th}}$ column, associated with an index belonging to the second group of indices θ_1 differing among themselves by positive integers; and so on through all the indices as far as θ_1^h.

A similar treatment is now applied to the columns $(h+1) \ldots (h+k)$, for which $\theta_r^{h+1} = \theta_r^{h+2} \ldots = \theta_r^{h+k}$ $(r = p \ldots 2)$; θ_1^{h+1} is given as the root of an equation of degree k; and the minors of the determinants χ_r, whose diagonals are $\theta_r^{h+1} \ldots \theta_r^{h+k}$, have the elements to the right of the diagonal suitably adjusted as above, while one equation of condition is furnished in connection with every difference $\theta_1^{h+r} - \theta_1^{h+s}$, which is an integer.

Supposing these equations all satisfied, consider the expansion of the matrix $\Omega\left(\frac{\chi_{p+1}}{t^{p+1}} + \ldots + \frac{\chi_1}{t}\right)$, which is effected in just the same way as for $p = 1$ (p. 22).

If $w = \sum_1^{p+1} \frac{\theta_r}{t^r}$, where θ_r is a matrix made up simply of the diagonal terms $\theta_r^1 \ldots \theta_r^n$, the application of the equation

$$\Omega(w+\sigma) = \Omega(w)\,\Omega[\Omega^{-1}(w)\,\sigma\Omega(w)]$$

resolves the required matrix into a product of two, of which the first, $\Omega(w)$, consists simply of diagonal terms of which the r^{th} is

$$\frac{e^{-\frac{\theta^r_{p+1}}{pt^p}-\frac{\theta_p^{\,r}}{(p-1)t^{p-1}}\cdots-\frac{\theta_2^{\,r}}{t}}\,t^{\theta_1^{\,r}}}{e^{-\frac{\theta^r_{p+1}}{pt_0^{\,p}}}\quad\cdots\cdots\quad t_0^{\theta_1^{\,r}}}.$$

The second matrix has zero everywhere in and to the left of the diagonal; and, since in the matrices χ the element in the r^{th} row and s^{th} column was zero unless $\theta_m^{\,r} = \theta_m^{\,s}$ ($m = p+1, \ldots 2$), it contains no exponential expressions. It can therefore be completely integrated in finite terms, just as was done in the case $p = 1$ (p. 23).

15. A simple example may be appended of the application of the method to a particular system. Consider the equation

$$x^4y'' + x^2y'(2-x) + y(1-3x+x^2) = 0.$$

Putting $y_1 = y$, $y_2 = x^2y'$, we have the linear system

$$y' = \begin{pmatrix} 0, & \dfrac{1}{x^2} \\ -\dfrac{1}{x^2}+\dfrac{3}{x}-1, & -\dfrac{2}{x^2}+\dfrac{3}{x} \end{pmatrix} y$$

$$= \left[\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\frac{1}{x^2} + \begin{pmatrix} 0 & 0 \\ 3 & 3 \end{pmatrix}\frac{1}{x} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right] y$$

The characteristic equation is

$$\begin{vmatrix} -\rho, & 1 \\ -\rho, & -2-\rho \end{vmatrix} = 0 \quad \text{or} \quad (\rho+1)^2 = 0,$$

giving equal roots -1 for ρ.

With $\mu = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ the transformed system becomes

$$y' = \left[\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\frac{1}{x^2} + \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}\frac{1}{x} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right] y$$

$$= uy \text{ say.}$$

Considering now the subsidiary equations

$$\eta' = u\eta - \eta\left[\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\frac{1}{x^2} + \begin{pmatrix} \theta_1 & \lambda \\ 0 & \theta_2 \end{pmatrix}\frac{1}{x}\right], \text{ and let } \eta = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}.$$

Then the equations to be satisfied are

I. $x_2^{\,1} - \theta_1 = 0$.

II. (1) $x_2^{\,2} - (1+\theta_1)\,x_1^{\,1} = 0$ (2) $(2-\theta_1)\,x_2^{\,1} - 1 = 0$,

I. and II. (2) give $(\theta_1 - 1)^2 = 0$.

We take $\theta_1 = 1$ therefore, so that $x_2^{\,1} = 1$ and $x_2^{\,2} = 2x_1^{\,1}$.

Again

III. (1) $x_2^{\,3} - \theta_1\,x_1^{\,2} = 0$, (2) $(1-\theta_1)\,x_2^{\,2} - x_1^{\,1} = 0$,

of which (2) gives $x_1^{\,1} = 0$, so that $x_2^{\,2} = 0$.

Similarly $x_1^2 = x_2^3 = 0$, and so on.

The equations for the second column are

$$\text{I. (1) } y_2^1 - x_1^1 - \lambda = 0, \quad \text{(2) } 3 - \theta_2 - x_1^2 = 0,$$

(2) gives $\theta_2 = 2 = 1 + \theta_1$ and (1) gives $y_2^1 = \lambda$.

$$\text{II. (1) } y_2^2 - (1+\theta_2)\, y_1^1 - x_1^2 - \lambda\, x_1^1 = 0, \quad \text{(2) } (2-\theta_2)\, y_2^1 - x_2^2 - \lambda x_2^1 = 0,$$

(2) gives $\lambda = 0$ and (1) gives $y_2^2 = 2y_1^1$ and also $y_1^1 = 0$.

$$\text{III. (1) } y_2^3 - \theta_2 y_1^2 - x_1^3 - \lambda x_1^2 = 0, \quad \text{(2) } (1-\theta_2)\, y_2^2 - y_1^1 = 0,$$

so that $y_1^1 = 0$ and $y_2^2 = 0$.

It is easily seen that all the remaining terms vanish.

Thus the solution reducing at $x = x_0$ to the matrix unity is

$$y = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \Omega \left\{ \frac{1}{x^2} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right\} \begin{pmatrix} 1 & 0 \\ x_0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mathrm{U}\Omega \left\{ \mathrm{U}^{-1} \frac{1}{x^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathrm{U} \right\} \begin{pmatrix} 1 & 0 \\ -x_0 & 1 \end{pmatrix},$$

where

$$\mathrm{U} = \Omega \left\{ \frac{1}{x^2} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right\} = \frac{e^{\frac{1}{x}}\, x}{e^{\frac{1}{x_0}}\, x_0} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{x}{x_0} \end{pmatrix}.$$

Thus

$$y = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \frac{e^{\frac{1}{x}}\, x}{e^{\frac{1}{x_0}}\, x_0} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{x}{x_0} \end{pmatrix} \Omega \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{x_0}{x} \end{pmatrix} \frac{1}{x^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{x}{x_0} \end{pmatrix} \right\} \begin{pmatrix} 1 & 0 \\ -x_0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \frac{e^{\frac{1}{x}}\, x}{e^{\frac{1}{x_0}}\, x_0} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{x}{x_0} \end{pmatrix} \Omega \begin{pmatrix} 0 & \dfrac{1}{xx_0} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \frac{e^{\frac{1}{x}}\, x}{e^{\frac{1}{x_0}}\, x_0} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{x}{x_0} \end{pmatrix} \begin{pmatrix} 1 & \dfrac{1}{x_0} \log \dfrac{x}{x_0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_0 & 1 \end{pmatrix}$$

$$= \frac{e^{\frac{1}{x}}\, x}{e^{\frac{1}{x_0}}\, x_0} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{1}{x_0} \log \dfrac{x}{x_0} \\ 0 & \dfrac{x}{x_0} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_0 & 1 \end{pmatrix}.$$

16. The number of conditions found in the course of the analysis shows that the solution in this form—which may be called the "normal" form, by analogy with the name "normal integrals" of linear equations—is by no means always possible. As many writers have pointed out, there is a much more general type of solution than the normal series for the ordinary linear equation, in the form of a normal series in a new independent variable $x^{1/k}$, k being a positive integer (CAYLEY, HAMBURGER, FABRY, &c.).

The method developed in the foregoing is peculiarly adaptable to the investigation of these integrals, inasmuch as the transformation to a new independent variable is very simply effected.

If in the linear system

$$\frac{dy}{dt} = u(t)y \quad \text{we put} \quad t = \phi(z),$$

we have without any calculation the new system for y as a function of z

$$\frac{dy}{dz} = \phi'(z)\{u(\overline{\phi z})\}y.$$

Suppose now $\phi(z) = z^k$. Then the transformed system is

$$\frac{dy}{dz} = kz^{k-1}u(z^k)y.$$

If, then, the original system is of rank p, so that

$$u(t) = \frac{\alpha_{p+1}}{t^{p+1}} + \ldots + \frac{\alpha_1}{t} + \alpha_0 + \beta_1 t + \ldots,$$

the new system is

$$\frac{dy}{dz} = \left[\frac{k\alpha_{p+1}}{z^{kp+1}} + \frac{k\alpha_p}{z^{k(p-1)+1}} + \ldots + k\alpha_0 z^{k-1} + \ldots\right]y$$

and is of rank kp.

If, now, we were to put $z = k^{1/kp} \cdot z^1$, the form of the equation would be unchanged, while the coefficient of $z^{1/kp+1}$ in the right-hand member would become the original canonical matrix α_{p+1}. This is, however, not necessary, as the whole investigation could be carried out equally well if any constants whatever replaced the unities to the right of the diagonal in α_{p+1}.

It may now well happen that though all the equations of condition found for the general system are not satisfied, those associated with the new system are all satisfied, so that the new system possesses a solution in normal form. If this is so, the original system will be said to admit of a solution in subnormal form. In fact, an integer k can always be found such that this is so, owing to the vanishing of the coefficients of $z^{-kp+r}\{r = 0, 1\ldots(k-2)\}$.

In the first place, all the conditions arrived at from the equations $X_1, Y_1, \ldots$ will be satisfied (p. 13), for the coefficient of z^{-kp} is identically zero; in general, the left-hand members of $X_r, Y_r \ldots$ are rational integral functions of the elements of the matrices $A_{kp}, A_{kp-1}, \ldots, A_{kp-r+1}$, if A_m stands for the matrix multiplying z^{-m}, and contain no term independent of these elements.

Now the conditions found on p. 13 arise from the equations $X_1 \ldots X_{\epsilon_1-1}, Y_1 \ldots Y_{\epsilon_1-2}, \ldots$, and therefore involve the matrices $A_{kp}, \ldots, A_{kp-\epsilon_1+2}$. These conditions will therefore all be satisfied if $k \geqq \epsilon_1$. Similarly, the analogous conditions for the second group of equal terms in the diagonal of α_{p+1} will be satisfied if $k \geqq \epsilon_2$, and so on.

Consider first, as being simplest of explanation and as containing the essential features, the case in which all the roots of the characteristic equation are equal, so that $\epsilon_1 = n$. It will be shown that a subnormal form satisfying the equation certainly exists if $\alpha_p^{1n} \neq 0$ with $k = n$.

We know from the foregoing investigation that $\theta^1{}_{np}$ is given as the root of an equation of degree n, and that, if the roots of this equation are all different, no more conditions than those just mentioned as satisfied are necessary to ensure the existence of the subnormal solution. But in this case the equation for $\theta^1{}_{np}$ is particularly easy to construct. We have, in fact,

$$nx_1{}^2 - \theta^1{}_{np} = 0, \qquad x_1{}^3 = x_1{}^4 = \ldots = x_1{}^n = 0,$$

$$nx_2{}^3 - \theta^1{}_{np}x_1{}^2 = 0, \qquad x_2{}^4 = \ldots = x_2{}^n = 0,$$

$$nx_3{}^4 - \theta^1{}_{np}x_2{}^3 - \theta^1{}_{np-1}x_1{}^3 = 0, \qquad x_3{}^5 = \ldots = x_3{}^n = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$nx^n{}_{n-1} - \theta^1{}_{np}x_{n-2}{}^{n-1} - \theta^2{}_{np}x_{n-3}{}^{n-1}\ldots = 0,$$

$$-\theta^1{}_{np}x^n{}_{n-1} - \ldots + A^{1n}{}_{n(p-1)+1} = 0,$$

from which at once we have

$$-\frac{(\theta^1{}_{np})^n}{n^{n-1}} + A^{1n}{}_{n(p-1)+1} = 0, \qquad \text{or} \qquad \theta^1{}_{np} = \frac{n\{A^{1n}{}_{n(p-1)+1}\}^{\frac{1}{n}}}{n}.$$

Thus, unless $A^{1n}{}_{n(p-1)+1} = 0$, the values of $\theta^1{}_{np}$ are all different, and a solution in subnormal form is therefore possible, as stated above, with the independent variable changed to $x^{1/n}$.

If, however, $A^{1n}{}_{n(p-1)+1} = 0$, we have

$$\theta^1{}_{np} = \ldots = \theta^n{}_{np} = 0,$$

and it will be found that the same conditions are necessary between the constants $An(p-1)+1$ as were found previously (p. 13) between the constants α_p, $A_{n(p-1)+1}$ being the same as $n \,.\, \alpha_p$, *e.g.*, from the equations

$$nx_n{}^n + A^{1,n-1}{}_{n(p-1)+1} = 0, \quad A^{2,n}{}_{n(p-1)+1} - nx_n{}^n = 0,$$

we have

$$A^{2,n}{}_{n(p-1)+1} + A^{1,n-1}{}_{n(p-1)+1} = 0, \; i.e, \; \alpha_p{}^{2,n} + \alpha_p{}^{1,n-1} = 0, \text{ and so on.}$$

Consider now what happens when these conditions are not all satisfied. Suppose, for instance, $\alpha_p{}^{2,n} + \alpha_p{}^{1,n-1} \neq 0$. Let the original system be transformed by the change of variables

$$x = z^k, \quad k = n-1.$$

Then from the equations

$$k \,.\, z_1{}^2 - \theta^1{}_{kp} = 0,$$

$$k \,.\, z_2{}^3 - \theta^1{}_{kp}z_1{}^2 = 0, \ldots$$

$$k \,.\, z^n{}_{n-1} - \theta^1{}_{kp}z_{n-2}{}^{n-1} + A^{1,n-1}{}_{k(p-1)+1} = 0, \quad -\theta^1{}_{kp}z^n{}_{n-1} + A^{2,n}{}_{k(p-1)+1}z_1{}^2 = 0,$$

we obtain the equation for $\theta^1{}_{kp}$

$$\frac{(\theta^1{}_{kp})^n}{k^n} - \frac{\theta_{kp}}{k}\left(\mathrm{A}^{1,n-1}{}_{k(p-1)+1} + \mathrm{A}^{2,n}{}_{k(p-1)+1}\right) = 0,$$

and since the last written coefficient is not zero, the roots of this equation are all different, and therefore the transformed system possesses normal integrals free from logarithms.

Again, suppose $\alpha_p{}^{1,n} = 0$ and that $\alpha_p{}^{1,n-1} + \alpha_p{}^{2n}$ also vanishes; then the roots of this equation all become zero, and we find that the condition $\alpha_p{}^{1,n-2} + \alpha_p{}^{2,n-1} + \alpha_p{}^{3,n} = 0$ is also necessary. If this is not satisfied and the original system be transformed by $x = z^k$, $k = n-2$, the equation for $\theta^1{}_{kp}$ becomes

$$\left(\frac{\theta^1{}_{kp}}{k}\right)^n - \left(\frac{\theta^1{}_{kp}}{k}\right)^2 \left(\mathrm{A}^{1,n-2}{}_{p(k-2)+1} + \mathrm{A}^{2,n-1}{}_{p(k-2)+1} + \mathrm{A}^{3,n}{}_{p(k-2)+1}\right) = 0,$$

which has two zero roots and the rest all different. If the one condition necessitated by the equality of the two zero roots is satisfied, the solution is again found. If this condition is not satisfied, the transformation $z = \zeta^2$ effects what is required.

Suppose now all the conditions of p. 13 are satisfied. Then whatever value of k be taken, we have $\theta^r{}_{kp} = 0$, so that, as far as we have seen, the transformation does not render the solution any nearer.

We must, in fact, proceed to consider the further conditions for the case when the roots θ_p are equal (p. 13).

Suppose, for instance, the first of these conditions is not satisfied, then putting $k = n$, we shall have $\theta^1{}_{kp} = \ldots = \theta^n{}_{kp}$, the conditions then necessary before the determination of θ_{kp-1} will be satisfied, and we shall eventually obtain a binomial equation for θ_{kp-1} of degree n in which the constant term does not vanish; the roots of this equation being all different, the subnormal integral exists.

Thus we may go through all the equations of condition in turn.

In the more general case, where the roots of the equation for θ_p fall into more than one group of equal roots, the procedure is exactly similar.

For example, suppose that $\alpha_p{}^{1,\epsilon_1}$, $\alpha_p{}^{\epsilon_1+1,\epsilon_1+\epsilon_2}$, ... are all different from zero. Then the preceding work suggests that a solution may be found in which the first ϵ_1 rows proceed according to powers of t^{1/ϵ_1}, the next ϵ_2 according to powers of t^{1/ϵ_2}, and so on.

The whole would thus be of normal form with the variable $t^{1/k}$, k being the least common multiple of ϵ_1, ϵ_2....

In fact, if we change the independent variable to $t^{1/k}$, k having this value, the matrix $\mathrm{A}_{kp-\epsilon_1+1}$ is identically zero, and the indices $\theta^1{}_{kp} \ldots \theta^{\epsilon_1}{}_{kp}$ are the roots of $-(\theta_{kp})^{\epsilon_1} + 0 = 0$.

These roots are all equal, and the corresponding equations of condition are all satisfied owing to the vanishing of the matrices other than A_{kr+1}.

If, now, we form the equation for $\theta^1{}_{kp-1}$ we have

$$kx_2{}^2 - \theta^1{}_{kp-1} = 0, \quad kx_4{}^3 - \theta^1{}_{kp-1}x_2{}^2 = 0, \quad \ldots\ldots,$$

$$-\theta^1{}_{kp-1}x^n{}_{2\epsilon_1-2} + \mathrm{A}^{1\epsilon_1}{}_{kp-2\epsilon_1+1} = 0,$$

giving
$$\theta^1_{kp-1} = k\{A^{1\epsilon_1}_{kp-2\epsilon_1+1/k}\}^{\frac{1}{\epsilon_1}}.$$

If $\frac{k}{\epsilon_1} = 2$, $A^{1n}_{kp-2\epsilon_1+1} = k \cdot \alpha_p$, and therefore if $\alpha_p^{1\epsilon_1} \neq 0$, the ϵ_1 roots of this equation are all different. If, however, $\frac{k}{\epsilon_1} > 2$, $A_{kp-2\epsilon_1+1}$ again is identically zero, and the necessary equations of condition are again satisfied.

Proceeding thus we find, in fact, that if $\alpha_p^{1\epsilon_1} \neq 0$, $\theta^s_{kp}, \ldots, \theta^s_{kp-k/\epsilon_1+2}$; $s = 1\ldots\epsilon_1$ all vanish and that $\theta^s_{kp-k/\epsilon_1+1}$ is the root of a binomial equation of degree ϵ_1, whose roots are all different, and so for the other divisors of k. If, however, $\alpha_p^{1\epsilon_1} = 0$ we have the same equations of condition again necessary, viz., $\alpha_p^{2\epsilon_1}$ and $\alpha_p^{1,\epsilon_1-1} = 0$, &c.

Assuming, then, that $\alpha_p^{1\epsilon_1} \neq 0$, we find, without difficulty, that all the quantities θ_r^s vanish for $s = 1\ldots\epsilon_1$, save those for which r is of the form $-k(p-m/\epsilon_1)+1$, so that the exponential arising in the first ϵ_1 rows involves only t^{1/ϵ_1} and not $t^{1/k}$.

The discussion of whether the solution of the subsidiary equation proceeds according to powers of t^{1/ϵ_1} only in the first ϵ_1 columns will not be carried out in full here. It is enough to know that, provided α_p^{1,ϵ_1}, $\alpha_p^{\epsilon_1+1,\epsilon_1+\epsilon_2}$ do not vanish, a subnormal form certainly exists satisfying the equation.

If, however, one or more of these quantities does vanish, and one of the consequent equations of condition is not satisfied, we may, as on pp. 31–33, find a new integer k, such that the necessary conditions for the existence of the subnormal form are satisfied.

17. As a concluding example consider the system derived from the equation of third order and rank one
$$y''' + \frac{a_{20}z^2 + a_{21}z + a_{22}}{z^4}y' + \frac{a_{30}z^3 + a_{31}z^2 + a_{32}z + a_{33}}{z^6}y = 0,$$
which with
$$y_1 = y, \quad y_2 = z^2y', \quad y_3 = z^4y'',$$
gives
$$y' = \left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_{33} & -a_{22} & 0 \end{pmatrix}\frac{1}{z^2} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ -a_{32} & -a_{21} & 4 \end{pmatrix}\frac{1}{z} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_{31} & -a_{20} & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_{30} & 0 & 0 \end{pmatrix}\right\}y.$$

The characteristic equation is $-\rho^3 - \rho a_{22} + a_{33} = 0$.

We shall confine ourselves to the case in which this equation has three equal roots. These must all then be zero, and $a_{22} = 0$, $a_{33} = 0$.

For the equation then to possess a normal solution we must have $a_{21} = 0$, $a_{32} = 0$. Supposing these conditions not satisfied, put $z = t^3$; then the equation becomes
$$y' = 3\left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{t^4} + \frac{0}{t^3} + \frac{0}{t^2} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ -a_{32} & -a_{21} & 4 \end{pmatrix}\frac{1}{t} + \ldots\right\}y.$$

The subsidiary equations then become
$$3x_1^2 - \theta_3^1 = 0, \quad x_1^3 = 0;$$
$$3x_2^2 - \theta_3^1x_1^1 - \theta_2^1 = 0, \quad 3x_2^3 - \theta_3^1x_1^2 = 0;$$
$$3x_3^2 - \theta_3^1x_2^1 - \theta_2^1x_1^1 - \theta_1^1 = 0, \quad 3x_3^3 - \theta_3^1x_2^2 - \theta_2^1x_1^2 = 0, \quad -\theta_3^1x_2^3 - \theta_2^1x_1^3 - 3a_{32} = 0.$$

The last gives $-\frac{1}{9}(\theta_3^1)^3 - 3a_{32} = 0$, so that $\theta_3^1 = -3(a_{32})^{\frac{1}{3}}$, where any cube root may be taken, the other roots giving θ_3^2, θ_3^3.

Further

$$3x_4^2 - \theta_3^1 x_3^1 - \theta_2^1 x_2^1 - (1 + \theta_1^1) x_1^1 = 0,$$

$$3x_4^3 - \theta_3^1 x_3^2 - \theta_2^1 x_2^2 + (1 - \theta_1^1) x_1^2 = 0,$$

$$-\theta_3^1 x_3^3 - \theta_2^1 x_2^3 + (3 - \theta_1^1) x_1^3 - 3a_{32} x_1^1 - 3a_{21} x_1^2 = 0\,;$$

of which the last gives

$$-\theta_2^1 \cdot \tfrac{3}{9}(\theta_3^1)^2 - a_{21}\theta_3^1 = 0,$$

so that $\theta_2^1 = a_{21}/(a_{32})^{\frac{1}{3}}$, and θ_2^2, θ_2^3 are given by taking the other roots for $(a_{32})^{\frac{1}{3}}$.

Lastly the equation

$$-\theta_3^1 x_4^3 - \theta_2^1 x_3^3 + (2 - \theta_1^1) x_2^3 - 3a_{32} x_2^1 - 3a_{21} x_2^2 = 0$$

gives

$$-\theta_3^1\left(\frac{\theta_3^1}{3} x_2^3 + \frac{\theta_2^1}{3} x_2^2 + \frac{1-\theta_1^1}{3} x_1^2\right) - \theta_2^1\left(\frac{\theta_3^1}{3} x_2^2 + \frac{\theta_2^1}{3} x_1^2\right) + (2 - \theta_1^1)\frac{\theta_3^1 x_1^2}{3} - 3a_{32} x_2^1 - 3a_{21} x_2^2 = 0,$$

which gives

$$\theta_1^1\left\{-\frac{(\theta_3^1)^2}{9}\right\} + \frac{5(\theta_3^1)^2}{9} - \frac{(\theta_2^1)^2\,\theta_3^1}{3} - a_{21}\theta_2^1 = 0,$$

so that $\theta_1^1 = 5$. Similarly, $\theta_1^2 = \theta_1^3 = 5$, and a subnormal form exists satisfying the equation, of which the first column has the determining factor $e^{\frac{3a_{32}^{\frac{1}{3}}}{2z^{\frac{2}{3}}} - \frac{a_{21}}{a_{32}^{\frac{1}{3}} z^{\frac{1}{3}}}} z^{\frac{5}{3}}$, and the other columns have the same factor with the other cube roots of a_{32}.

We may remark that this agrees with the results obtained for this equation by the ordinary methods (FORSYTH, "Linear Differential Equations," § 99) under the assumption that $a_{32} \neq 0$. We have shown that this is a necessary condition for the existence of the subnormal form in the variable $t = z^{\frac{1}{3}}$ satisfying the equation formally, unless we have also $a_{21} = 0$.

If, however, $a_{32} = 0$ and $a_{21} \neq 0$, then, as we have seen above, the transformation $t = z^{\frac{1}{2}}$ will give us a system admitting of 3 normal solutions; the equation for θ_1 is, in fact, $(\theta_1)^3 - 4a_{21}\theta_1 = 0$, giving $\theta_1 = 0$ or $\pm 2a_{21}^{\frac{1}{2}}$.

We see, in fact, that, when $a_{32} = 0$ and $a_{21} \neq 0$, the characteristic index of the original equation is 2, so that there will be one regular form satisfying the equation, *i.e.*, an expression of the form $x^\rho P(x)$.

If a_{32}, a_{21} are both zero, the equation is of Fuchsian type. Thus the normal or subnormal forms are found in all cases.